Cyclotomic Hecke algebras: Jucys-Murphy elements, representations, classical limit

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Abstract

An inductive approach to the representation theory of cyclotomic Hecke algebras, inspired by Okounkov and Vershik [28], is developed. We study the common spectrum of the Jucys-Murphy elements using representations of the simplest affine Hecke algebra. Representations are constructed with the help of a new associative algebra whose underlying vector space is the tensor product of the cyclotomic Hecke algebra with the free associative algebra generated by standard m-tableaux. The classical limit of the whole approach, including the construction of representations, is given. The flatness of the deformation is proved without the use of the representation theory.

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Contents

1. Introduction

The A-type Hecke algebra $H_n(q)$ is a one-parameter deformation of the group ring of the symmetric group S_n. We shall often omit the reference to the deformation parameter q and write simply H_n (for other families of algebras appearing in this article the reference to the parameters of the family will also be often omitted in the notation). The Hecke algebra of type A plays an important role in numerous subjects: we just mention the knot theory, the Schur–Weyl duality for the quantum general linear group; the representation theory of the A-type Hecke algebra for q a root of unity (for generic q the Hecke algebra is isomorphic to the group ring of S_n) is related to the modular representation theory of the symmetric group.

The Hecke algebras H_n form, with respect to n, an ascending chain of algebras. The Hecke algebras H_n possess a set of Jucys–Murphy elements. This is a maximal commutative set (for generic q and in the classical limit) whose advantages are: explicit description (compared to other maximal commutative sets discovered in the study of chain models); simple relation to the centralizers of the members of the chain. Moreover, the inductive formula for the Jucys–Murphy elements can be lifted to the "universal" level: there exists a chain of affine Hecke algebras $\hat{H}_n$ for which one also defines the Jucys–Murphy elements; the subalgebra generated by the Jucys–Murphy elements is isomorphic to a free commutative algebra and there is a surjection $\hat{H}_n \to H_n$ which sends the Jucys–Murphy elements of the affine Hecke algebra to the Jucys–Murphy elements of the Hecke algebra H_n.

The main relation participating in the definition of the Hecke algebras is the Artin (or braid or Yang–Baxter) relation. The main additional relation participating in the definition of the affine Hecke algebra and governing the commutativity of the set of the Jucys–Murphy elements is the reflection equation (a sort of braid relation participating in the presentation of the Weyl groups of type B) . The Jucys–Murphy elements and reflection equation appear in many applications: scattering on the half-line [5], theory of knots in a torus (see, *e.g.*, [21] and references therein), the standard complex and the BRST operators for quantum Lie algebras [11, 17], matrix integrals [43], quantum multilinear algebra [18], construction of the quantum Minkowski space [26, 8] and quantum versions of the accidental isomorphisms [19] *etc.*

In [28] Okounkov and Vershik developed an inductive approach to the representation theory of the chain of the symmetric groups. Within this approach the description of the irreducible representations, the Young graph and the Young orthogonal form arise from the study of the spectrum of the set of the

Jucys–Murphy elements of the group ring $\mathbb{C}S_n$. This approach has then been successfully generalized to the projective representations of the symmetric group [34], to the wreath product $G \wr S_n$ of the symmetric group with any finite group G [29], to the Hecke algebras of type A [14] and to the Birman-Murakami-Wenzl algebras of type A [15].

An analogue of the Hecke algebra and of the braid group exists for all complex reflection groups. The complex reflection groups generalize the Coxeter groups and the complete list of irreducible finite complex reflection groups consists of the series of groups denoted $G(m,p,n)$, where m,p,n are positive integers such that p divides m, and 34 exceptional groups [33].

The Hecke algebra of $G(m,1,n)$, which we denote by $H(m,1,n)$, is our main object of study. The algebra $H(m,1,n)$ has been introduced in [1, 6] and is called the cyclotomic Hecke algebra. For $m=1$ this is the Hecke algebra of type A and for $m=2$ this is the Hecke algebra of type B. The representation theory of the algebra $H(m,1,n)$ was developed in [1] (and in [13] for the Hecke algebra of type B); the irreducible representations of the algebra $H(m,1,n)$ are labeled, as for the group $G(m,1,n)$, by m-tuples of Young diagrams.

The Hecke algebras $H(m,1,n)$ also (as the usual Hecke algebras) form, with respect to n, an ascending chain of algebras. Moreover, the algebra $H(m,1,n)$ is naturally a quotient of the affine Hecke algebra $\hat{H}_n$ and therefore inherits the natural set of Jucys–Murphy elements from the affine Hecke algebra. Our principal aim in this paper is to reproduce the representation theory of the cyclotomic Hecke algebra by analyzing the spectrum of Jucys–Murphy operators. We generalize the Okounkov-Vershik approach to the representation theory of the cyclotomic Hecke algebra $H(m,1,n)$; we construct irreducible representations and show that the usage of this approach allows to describe all irreducible representations of $H(m,1,n)$ upon certain restrictions (slightly stronger than the semi-simplicity conditions) on the parameters of the algebra $H(m,1,n)$.

We stress that our aim is not the construction itself of the representation theory - it has been already constructed in [1] - but we want to re-obtain the representations directly from the analysis of the Jucys–Murphy operators, to encode the representation bases in terms of sets of numbers which satisfy simple rules and which in fact are sets of common eigenvalues of the Jucys–Murphy elements and to reinterpret the Young multi-tableaux in terms of strings of eigenvalues of the Jucys–Murphy operators. As a byproduct of the construction of the representation bases it follows that the set of Jucys–Murphy operators is *maximal* commutative (this observation is present in [1]). The approach, based on the Jucys–Murphy operators, has a recursive nature - it uses the structure of the ascending, with respect to n, chain of the cyclotomic algebras $H(m,1,n)$.

Since the cyclotomic Hecke algebra is the quotient of the affine Hecke algebra, a representation of the cyclotomic Hecke algebra is also a representation of the affine Hecke algebra. The representations of the affine Hecke algebras are usually expressed in a different language, see [4, 42, 32, 37, 3] for the original papers and surveys of the classical and q-deformed situations.

A novelty of our construction of the representations of the cyclotomic Hecke algebra is the introduction of a new associative algebra. Namely we equip with an algebra structure the tensor product of the algebra $H(m,1,n)$ with a free associative algebra generated by the standard m-tableaux corresponding to a given m-partition of n. We denote the resulting algebra by $\mathfrak{T}$. The representations are built then by evaluation from the right with the help of the simplest one-dimensional representation of $H(m,1,n)$. There is a natural "evaluation" homomorphism from $\mathfrak{T}$ to $H(m,1,n)$ sending the generator labeled by a standard m-tableau to the corresponding primitive idempotent of the algebra $H(m,1,n)$.

An interesting consequence of the existence of this "smash" product with the free algebra is a structure of a module on the tensor product of two representations corresponding to two (in general, any number of) m-partitions. We determine the rules of the decomposition of these tensor products into direct sums of irreducible representations. The decomposition rules themselves are quite easy: $V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}}\cong \dim(V_{\lambda'^{(m)}})\,V_{\lambda^{(m)}}$; however an intertwiner, establishing this isomorphism, is difficult to describe explicitly - already for small n the simplest choice of an intertwiner is quite evolved. Also, to obtain the decomposition rules we need the completeness result saying that every $H(m,1,n)$-module is isomorphic to a direct sum of $H(m,1,n)$-modules corresponding to m-partitions of n; however the definition of the module structure on the tensor product does not use the completeness. We think that the module structure on the tensor product deserves further study.

The space of a representation obtained with the help of the smash product is equipped with a distinguished basis which is analogous to the semi-normal basis for representations of the symmetric group. It turns out that there exists an analogue of an invariant scalar product on the representation spaces; the definition of this analogue involves the involution $\omega\colon\ q\to q^{-1}$, $v_j\to v_j^{-1}$, $j=1,\dots,m$ (q and v_j, $j=1,\dots,m$, are the parameters of the algebra $H(m,1,n)$, see Section 2 for precise definitions). We call this analogue "ω-sesquilinear scalar product" and compute it for all irreducible representations of $H(m,1,n)$ building thereby the analogues of the orthogonal representations of the symmetric group. As a consequence, there is a large class of finite-dimensional irreducible representations of the affine Hecke algebra which are unitarizable with respect to this scalar product. If the parameters q and v_j, $j=1,\dots,m$, of the algebra $H(m,1,n)$ take values on the unit circle (in $\mathbb{C}$) then the ω-sesquilinear scalar product becomes a usual Hermitian scalar product.

On the level of groups the Coxeter–Todd algorithm [7] is a powerful method for constructing a normal form for the group elements (with respect to a given subgroup). For an ascending chain of groups the Coxeter–Todd algorithm provides (recursively) a global normal form for the group elements. We apply the Coxeter–Todd algorithm to the chain, with respect to n, of the groups $G(m,1,n)$. The resulting normal form has a nice generalization providing a normal form for the elements of the cyclotomic algebra $H(m,1,n)$ (in other words, a basis of $H(m,1,n)$). Our basis is quite different from the basis $\mathfrak{B}_{\mathrm{AK}}$ suggested in [1].

The cyclotomic Hecke algebra $H(m,1,n)$ is a deformation of the group ring of the group $G(m,1,n)$. The first advantage of our basis is that it allows to prove the flatness of the deformation without any appeal to the representation theory (the flatness of the deformation was established in [1] as an outcome of the representation theory, after the classification of irreducible representations of the algebra $H(m,1,n)$; note that for the cyclotomic quotient of the degenerate affine Hecke algebra a proof that the elements from $\mathfrak{B}_{\mathrm{AK}}$ form a basis is given in [20] without the use of the representation theory).

Another advantage of our basis which underlines its naturality is that it is well-adapted to the structure of the chain of the cyclotomic algebras; in particular, the formulas for the induced representations (from the algebra $H(m,1,n-1)$ to the algebra $H(m,1,n)$) are quite easy and we write them down explicitly. A representation of the algebra $H(m,1,n)$ induced from a one-dimensional representation of the algebra $H(m,1,n-1)$ is a natural analogue of the Burau representation.

The inductive approach to the representation theory of the usual, type A, Hecke algebras heavily uses the representation theory of the affine Hecke algebra of type A. One could expect that there will be a necessity to use representations of the affine Hecke algebra of type B (see [9, 23] for definitions) for the representation theory of the cyclotomic Hecke algebras. But – and it is maybe surprising – in the non-degenerate situation the representation theory of the Hecke algebra in the inductive approach requires the study of representations of the same affine Hecke algebra of type A. However, in the classical limit, a certain version of the degenerate affine cyclotomic Hecke algebras, which we denote $\mathfrak{A}_{m,n}$ in the text, appears; the representations of the simplest degenerate affine cyclotomic Hecke algebra $\mathfrak{A}_{m,2}$ serve for the study of the representation theory of the corresponding complex reflection group – the classical limit of the cyclotomic Hecke algebra.

The algebras $\mathfrak{A}_{m,n}$ for all $m=1,2,\dots$ can be obtained by a certain limiting procedure from one and the same affine Hecke algebra $\hat{H}_n$.

The representation theory of $G(2,1,n)$ (the Coxeter group of type B) was studied by Young [41] and the representation theory of the wreath product $A \wr S_n$ of an arbitrary finite group A by the symmetric group (of which $G(m,1,n)$ is a particular case) was studied in [35]. Given a finite group A the wreath products $A \wr S_n$ form, with respect to n, an ascending chain of groups. The Okounkov-Vershik approach has been extended to the representation theory of the wreath product of an arbitrary finite group by the symmetric group in [29]. The branching rules for the chain of groups $A \wr S_n$ are multiplicity free if and only if the group A is abelian. The chain of groups $G(m,1,n)$ provides a simplest example (here A is a cyclic group) of a multiplicity free chain of wreath products $A \wr S_n$.

In this paper, we attentively describe the classical limit of the whole construction including the smash product, this time of the group ring of $G(m,1,n)$, with the free associative algebra generated by the standard m-tableaux. Some parts of the construction turn out to be more complicated than in the non-degenerate situation. As a rule, we omit the proofs of statements in the classical limit when they almost repeat the proofs of the corresponding statements for the cyclotomic Hecke algebras. But when the degenerate picture does not exactly follow the non-degenerate one, we attempt to provide the full information. It concerns, in particular, the structure of the degenerate affine cyclotomic Hecke algebras and commutative sets in them and also the subtleties about the intertwining operators.

The representations obtained with the help of the smash product are analogues, for $G(m,1,n)$, of the semi-normal representations of the symmetric group. We determine the $G(m,1,n)$-invariant Hermitian scalar product on the representations and describe analogues of the orthogonal representations of the symmetric group.

Some of results, concerning representations of the cyclotomic Hecke algebras, smash product with free algebras, normal form *etc.*, of this paper were announced in [27] without proofs. We provide all necessary proofs here.

1.1 Organization of the paper

In Section 2 we recall the definitions of various chains of groups and algebras featuring in this article, and of the Jucys–Murphy elements of the chain of the braid groups and chains of quotients of the braid group ring.

Sections 3, 4 and 5 are devoted to the representation theory of the non-degenerate cyclotomic Hecke algebras.

In Section 3 we start the study of the representation theory of the chain, with respect to n, of the cyclotomic Hecke algebras $H(m,1,n)$ generalizing Okounkov-Vershik approach of the representation theory of the symmetric groups. An important tool here is the list of representations, satisfying some natural properties, of the affine Hecke algebra $\hat{H}_2$. We relate the set of standard Young m-tableaux with the set $\mathrm{Spec}(J_1,\dots,J_n)$ of common eigenvalues of the Jucys–Murphy elements $J_1,\dots,J_n$ in a certain class of representations (which we call C-representations) of $H(m,1,n)$. More precisely, we show that any string of numbers belonging to $\mathrm{Spec}(J_1,\dots,J_n)$ is contained in a set called $\mathrm{Cont}(n)$ which is in bijection with the set of standard Young m-tableaux.

In Section 4 we equip, for any m-partition $\lambda^{(m)}$ of n, the space $\mathbb{C}[\mathcal{X}_{\lambda^{(m)}}]\otimes H(m,1,n)$ with a structure of an associative algebra. Here $\mathbb{C}[\mathcal{X}_{\lambda^{(m)}}]$ is the free associative algebra with generators $\mathcal{X}_{\lambda^{(m)}}$ labeled by the standard m-tableaux of shape $\lambda^{(m)}$. To define the algebra structure, it is convenient to use the Baxterized form of (only) a part of generators of the cyclotomic Hecke algebra. Given a one-dimensional representation of the algebra $H(m,1,n)$, we construct, with the help of the algebra structure on $\mathbb{C}[\mathcal{X}_{\lambda^{(m)}}]\otimes H(m,1,n)$, a representation on the space whose basis is labeled by the standard m-tableaux of shape $\lambda^{(m)}$. This construction implies that the set $\mathrm{Spec}(J_1,\dots,J_n)$ of common eigenvalues of the Jucys–Murphy elements in C-representations coincides with the set $\mathrm{Cont}_m(n)$ and with the set of standard Young m-tableaux.

At the end of Section 4 we compute the invariant ω-sesquilinear scalar product for all representations $V_{\lambda^{(m)}}$.

Also, in Appendix to Section 4 we explain how the algebra structure on the space $\mathbb{C}[\mathcal{X}_{\lambda^{(m)}}]\otimes H(m,1,n)$ induces the tensor structure on the set of C-representations of the cyclotomic Hecke algebra $H(m,1,n)$; more generally, given an $H(m,1,n)$-module W, the algebra structure on $\mathbb{C}[\mathcal{X}_{\lambda^{(m)}}]\otimes H(m,1,n)$ leads to a structure of an $H(m,1,n)$-module on the space $V_{\lambda^{(m)}}\otimes W$, where $V_{\lambda^{(m)}}$ is the C-representation related to the m-partition $\lambda^{(m)}$. We determine the rules of the decomposition of the tensor products of C-representations into direct sums of irreducible representations. In the course of the proof we give several explicit examples of such decompositions.

In Section 5 we complete the representation theory of the cyclotomic Hecke algebras; we show that the constructed representations are irreducible and pairwise non-isomorphic (the proof is included for completeness; it is adopted from [1]). Using an upper bound (proved in Appendix A) for the dimension of the cyclotomic Hecke algebra and some results about products of Bratteli diagrams recalled in Appendix B we conclude in a standard way that the class of irreducible C-representations exhausts the set of the irreducible representations of the cyclotomic Hecke algebra when the parameters of the algebra satisfy the restrictions specified in Section 2.

Further, we include in Section 5 several direct consequences of the developed representation theory (valid either in the generic picture or under the restrictions on the parameters considered in this article): the semi-simplicity of the cyclotomic Hecke algebra $H(m,1,n)$, the simplicity of the branching rules for the representations of the chain of the cyclotomic Hecke algebras[3] and the maximality in $H(m,1,n)$ of the commutative set formed by the Jucys–Murphy elements. We also mention some information, implied by the developed representation theory, about the structure of the centralizer of the algebra $H(m,1,n-1)$ considered as a subalgebra in $H(m,1,n)$.

The classical limit of the cyclotomic Hecke algebra $H(m,1,n)$ is the group ring of the complex reflection group $G(m,1,n)$. Section 6 is entirely devoted to the group $G(m,1,n)$.

The representation theory of $H(m,1,n)$ developed in Sections 3, 4 and 5 can be used to immediately obtain the representation theory of the group $G(m,1,n)$: one only has to take the classical limit of the parameters in the formulas for the matrix elements of the generators. Nevertheless it is illuminating to build the representation theory of the chain of groups independently of the non-degenerate picture. We present the classical limit of the whole construction developed in Sections 3, 4 and 5 establishing the inductive approach to the representation theory of the chain, with respect to n, of the groups $G(m,1,n)$.

We first explain how to obtain the classical Jucys–Murphy elements of the group ring $\mathbb{C}G(m,1,n)$ from the Jucys–Murphy elements of the non-degenerate cyclotomic Hecke algebra $H(m,1,n)$ (thus extending a result in [30] on the Weyl groups and their Hecke algebras). As for the relation with the affine Hecke algebra, the picture complicates on the classical level; we introduce a degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$. The degenerate cyclotomic affine Hecke algebras also form a chain with respect to n. We establish the commutativity of a certain set of elements of the algebra $\mathfrak{A}_{m,n}$; the elements of this commutative set we call "classical universal" Jucys–Murphy elements. There is a surjection $\mathfrak{A}_{m,n} \to \mathbb{C}G(m,1,n)$ and the classical Jucys–Murphy elements are the images of the classical universal Jucys–Murphy elements of $\mathfrak{A}_{m,n}$ under this surjection; we obtain therefore an independent of the non-degenerate picture proof of the commutativity of the set formed by the classical Jucys–Murphy elements.

Then we grosso modo repeat the same steps as in the non-degenerate situation. We study a certain class of irreducible representations of the algebra $\mathfrak{A}_{m,2}$ and deduce that the spectrum of the classical Jucys–Murphy elements in C-representations is included in a set $\mathrm{cCont}_m(n)$ which is in bijection with the set of the standard Young m-tableaux. We introduce an algebra structure on the tensor product of the algebra $\mathbb{C}G(m,1,n)$ with a free associative algebra generated by the standard m-tableaux. We sketch the analogue of the construction of representations of $H(m,1,n)$ in the classical picture and conclude that the constructed representations exhaust the set of irreducible representations of the group $G(m,1,n)$. We do not give the proofs of the statements concerning the group $G(m,1,n)$ when they go along the same lines as in the non-degenerate situation; we only indicate modifications when

[3]The paper [12] advocates the point of view that it is the affine Hecke algebra "that is responsible for the multiplicity one phenomena"; for generic finite-dimensional representations the multiplicity one statement for the affine Hecke algebra of type A follows from the multiplicity one statement for the cyclotomic Hecke algebra because in a generic finite-dimensional representation the spectrum of the Jucys–Murphy element J_1 is finite and thus such representation is actually a representation of a cyclotomic quotient of the affine Hecke algebra.

they appear.

Section 6 contains two appendices. The first appendix deals with the structure theory for the degenerate cyclotomic affine Hecke algebras $\mathfrak{A}_{m,n}$; we build a basis of $\mathfrak{A}_{m,n}$ providing a normal form for the elements of $\mathfrak{A}_{m,n}$. In the second appendix we study classical intertwining operators, useful in analyzing the spectrum of the classical Jucys–Murphy operators; we show how to obtain classical intertwining operators as classical limits of certain intertwining operators in the non-degenerate affine Hecke algebra $\hat{H}_n$.

Parts of the results in Section 6 can be found in the literature; we include them for the material to be self-contained. Namely, the Jucys-Murphy elements of the group ring of the wreath product of an arbitrary finite group A by the symmetric group (of which $\mathbb{C}G(m,1,n)$ is a particular example) have been defined independently in [29] and [40]; the Jucys–Murphy elements of the group ring of the Coxeter group $G(2,1,n)$ were introduced in [30]. In [29], the Okounkov-Vershik approach is extended to the wreath products of any finite group A by the symmetric group. Also, the degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$ turns out to coincide with the wreath Hecke algebra adapted to our situation; the wreath Hecke algebra was defined and studied in [39] (see also [38]). The algebra $\mathfrak{A}_{m,n}$, defined differently, also appears in [31] as an analogue for $G(m,1,n)$ of a graded Hecke algebra. In our presentation we especially insist on the connection of the treatment of the group $G(m,1,n)$ with our treatment of the cyclotomic Hecke algebra $H(m,1,n)$; in particular, as we said above, we obtain the Jucys-Murphy elements for $G(m,1,n)$ (respectively, the intertwining operators for $\mathfrak{A}_{m,n}$) by taking limits of certain expressions involving the Jucys-Murphy elements of $H(m,1,n)$ (respectively, the intertwining operators for $\hat{H}_n$).

The article concludes with Appendices A, B and C.

In Appendix A we present the Coxeter-Todd algorithm for the chain, with respect to n, of the groups $G(m,1,n)$. We already mentioned in Introduction and we partly repeat here several uses of the algorithm. The Coxeter-Todd algorithm provides a normal form for elements of the groups $G(m,1,n)$. We establish a generalization of this normal form to the cyclotomic Hecke algebras $H(m,1,n)$. The normal form allows us in particular to prove, without the use of representation theory, that the deformation from $\mathbb{C}G(m,1,n)$ to $H(m,1,n)$ is flat in the sense that the dimension of $H(m,1,n)$ equals the order of $G(m,1,n)$. In the course of building of the normal form for elements of $H(m,1,n)$ we obtain explicit formulas for induced representations of $H(m,1,n)$ with respect to $H(m,1,n-1)$. At the end of Appendix A we specify these formulas to the usual Hecke algebras and extend these formulas to the affine Hecke algebras. All these results are placed in a separate Appendix since they are not directly related to the representation theory of the cyclotomic Hecke algebras – for Section 5 we do not need the precise statement about the flatness of the deformation; we need only an upper bound for the dimension of the algebra; the assertion about the upper bound, see the first Proposition in Subsection 2 of Appendix A, constitutes the simple part of the flatness statement. The subject of Appendix A is rather the structure theory of the cyclotomic Hecke algebras.

In Appendix B we recall some definitions and results concerning Bratteli diagrams and their products. We specify the information to the powers of the Young graph; the m-th power of the Young graph is relevant in the representation theory of the chain of the cyclotomic Hecke algebras. Appendix B has a review character.

Appendix C contains several examples of the defining relations of the algebra on the tensor product of the algebra $H(m,1,n)$ with a free associative algebra generated by the standard m-tableaux; resulting explicit formulas for matrix elements of generators in low-dimensional irreducible representations of the cyclotomic Hecke algebras $H(m,1,n)$ are given.

Notation.

In this article, the ground field is the field $\mathbb{C}$ of complex numbers.

The spectrum of an operator $\mathcal{T}$ is denoted by $\mathrm{Spec}(\mathcal{T})$.

We denote, for two integers $k,l\in\mathbb{Z}$ with $k<l$, by $[k,l]$ the set of integers $\{k,k+1,\dots,l-1,l\}$.

The q-number j_q is defined by $j_q := \dfrac{q^j-q^{-j}}{q-q^{-1}}$.

The diagonal matrix with entries $z_1,z_2,\dots,z_k$ (on the diagonal) is denoted by $\mathrm{diag}(z_1,z_2,\dots,z_k)$.

2. Cyclotomic Hecke algebras and Jucys–Murphy elements

The braid group B_n of type A (or simply the braid group) on n strands is generated by the elements $\sigma_1,\dots,\sigma_{n-1}$ with the defining relations:

$$\begin{cases}\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } i=1,\dots,n-2\ , \qquad (2.1)\\ \sigma_i\sigma_j=\sigma_j\sigma_i & \text{for all } i,j=1,\dots,n-1 \text{ such that } |i-j|>1\ . \qquad (2.2)\end{cases}$$

The braid group αB_n of type B (sometimes called *affine* braid group) is obtained by adding to the generators $\sigma_1,\dots,\sigma_{n-1}$ the generator τ with the defining relations (2.1), (2.2) and:

$$\begin{cases}\tau\sigma_1\tau\sigma_1=\sigma_1\tau\sigma_1\tau\ , & \qquad (2.3)\\ \tau\sigma_i=\sigma_i\tau & \text{for } i>1\ . \qquad (2.4)\end{cases}$$

The elements J_i, $i=1,\dots,n$, of the braid group of type B defined inductively by the following initial condition and recursion:

$$J_1=\tau\ ,\quad J_{i+1}=\sigma_iJ_i\sigma_i\ , \tag{2.5}$$

are called Jucys–Murphy elements. It is well known that they form a commutative set of elements. In addition, J_i commutes with all σ_k except σ_{i-1} and σ_i,

$$J_i\sigma_k=\sigma_kJ_i \quad \text{if} \quad k>i \quad \text{or} \quad k<i-1\ . \tag{2.6}$$

The affine Hecke algebra $\hat{H}_n$ is the quotient of the group algebra of the B-type braid group αB_n by:

$$\sigma_i^2=(q-q^{-1})\sigma_i+1 \qquad \text{for all } i=1,\dots,n-1\ . \tag{2.7}$$

The usual Hecke algebra H_n is the algebra generated by the elements $\sigma_1,\dots,\sigma_{n-1}$ with the relations (2.1)–(2.2) and (2.7).

The cyclotomic Hecke algebra $H(m,1,n)$ is the quotient of the affine Hecke algebra $\hat{H}_n$ by

$$(\tau - v_1)\dots(\tau - v_m) = 0\ . \tag{2.8}$$

In particular, $H(1,1,n)$ is isomorphic to the Hecke algebra of type A and $H(2,1,n)$ is isomorphic to the Hecke algebra of type B.

The algebra $H(m,1,n)$ is a deformation of the group algebra $\mathbb{C}G(m,1,n)$ of the complex reflection group $G(m,1,n)$. The group $G(m,1,n)$ is isomorphic to $S_n \wr C_m$, the wreath product of the cyclic group with m elements by the symmetric group S_n. We collected the definition and some properties of the group $G(m,1,n)$ in Section 6.

The deformation from $\mathbb{C}G(m,1,n)$ to $H(m,1,n)$ is flat in the sense that $H(m,1,n)$ is a free $\mathbb{C}[q,q^{-1},v_1,\dots,v_m]$-module of dimension equal to the order of $G(m,1,n)$, that is:

$$\dim(H(m,1,n)) = n!\cdot m^n\ . \tag{2.9}$$

The flatness is proved in [1] with the help of the representation theory and in Appendix A of the present article within the theory of associative algebras.

The specialization of $H(m,1,n)$ is semi-simple if and only if the numerical values of the parameters satisfy (see [2])

$$1+q^2+\dots+q^{2N}\neq 0 \ \text{ for all }\ N:\ N<n \tag{2.10}$$

and

$$q^{2i}v_j - v_k \neq 0 \ \text{ for all }\ i,j,k\ \text{ such that }\ j\neq k \text{ and } -n<i<n\ . \tag{2.11}$$

In the sequel we work either with a generic cyclotomic Hecke algebra (that is, v_1, …, v_m and q are indeterminates) or in the semi-simple situation with an additional requirement:

$$v_j \neq 0\ ,\ j=1,\dots,m\ . \tag{2.12}$$

As n varies, the algebras $H(m,1,n)$ form an ascending chain of algebras:

$$H(m,1,0)=\mathbb{C}\subset H(m,1,1)\subset\dots\subset H(m,1,n)\subset\dots \tag{2.13}$$

(the elements τ and $\sigma_1,\dots,\sigma_{n-2}$ of the algebra $H(m,1,n)$ generate a subalgebra isomorphic to $H(m,1,n-1)$). One has similar ascending chains of braid groups, affine braid groups and affine Hecke algebras.

The representation theory of the generic algebra $H(m,1,n)$ was studied in [1]. Here we present another approach which is a generalization of the approach of Okounkov and Vershik to the representation theory of the symmetric group [28] and which refers to the ascending chain (2.13).

We shall denote by the same symbols J_i the images of the Jucys–Murphy elements in the cyclotomic Hecke algebra. As a by-product of the representation theory of the generic algebra $H(m,1,n)$, the set of the Jucys–Murphy elements $\{J_1,\dots,J_n\}$ is maximal commutative in $H(m,1,n)$; more precisely, the algebra of polynomials in the Jucys–Murphy elements coincides with the algebra generated by the union of the centers of $H(m,1,k)$ for $k=1,\dots,n$.

Remark. We use the same notation "σ_i" for generators of different groups and algebras: these are braid and affine braid groups and Hecke, affine Hecke and cyclotomic Hecke algebras. The symbol τ is also used to denote a generator of several different objects. Moreover the reference to n (as in H_n) in our notation for the generators σ_i is also always omitted. This should not lead to any confusion, it will be clear from the context what is the algebra/group in question.

3. Spectrum of Jucys–Murphy elements and Young m-tableaux

We begin to develop an approach, based on the Jucys–Murphy elements, to the representation theory of the chain (with respect to n) of the cyclotomic Hecke algebras $H(m,1,n)$. This is a generalization of the approach of [28].

1. The first step consists in construction of all representations of $H(m,1,n)$ verifying two conditions. First, the Jucys–Murphy elements $J_1,\dots,J_n$ are represented by semi-simple (diagonalizable) operators. Second, for every $i=1,\dots,n-1$ the action of the subalgebra generated by J_i, J_{i+1} and σ_i is completely reducible. We shall use the name C-representations (C is the first letter in "completely reducible") for these representations. At the end of the construction we shall see that all irreducible representations of $H(m,1,n)$ are C-representations.

Following [28] we denote by $\mathrm{Spec}(J_1,\dots,J_n)$ the set of strings of eigenvalues of the Jucys–Murphy elements in the set of C-representations: $\Lambda=(a_1^{(\Lambda)},\dots,a_n^{(\Lambda)})$ belongs to $\mathrm{Spec}(J_1,\dots,J_n)$ if there is a vector e_Λ in the space of some C-representation such that $J_i(e_\Lambda)=a_i^{(\Lambda)}e_\Lambda$ for all $i=1,\dots,n$. Every C-representation possesses a basis formed by vectors e_Λ (this is a reformulation of the first condition in the definition of C-representations). Since σ_k commutes with J_i for $k>i$ and $k<i-1$, the action of σ_k on a vector e_Λ, $\Lambda\in\mathrm{Spec}(J_1,\dots,J_n)$, is "local" in the sense that $\sigma_k(e_\Lambda)$ is a linear combination of $e_{\Lambda'}$ such that $a_i^{(\Lambda')}=a_i^{(\Lambda)}$ for $i\neq k,k+1$.

2. Affine Hecke algebra $\hat H_2$. Consider the affine Hecke algebra $\hat H_2$, generated by X, Y and σ with the relations:

$$XY=YX,\quad Y=\sigma X\sigma,\quad \sigma^2=(q-q^{-1})\sigma+1\ . \tag{3.1}$$

For all $i=1,\dots,n-1$, the subalgebra of $H(m,1,n)$ generated by J_i, J_{i+1} and σ_i is a quotient of $\hat H_2$. We reproduce here the result of [14] concerning the classification of irreducible representations with diagonalizable X and Y of the algebra $\hat H_2$.

There are one-dimensional and two-dimensional irreducible representations.

- The one-dimensional irreducible representations are given by

$$X\mapsto a,\quad Y\mapsto q^{\pm 2}a,\quad \sigma\mapsto \pm q^{\pm 1}\ . \tag{3.2}$$

- The two-dimensional irreducible representations are given by

$$\sigma\mapsto\begin{pmatrix}0 & 1\\ 1 & q-q^{-1}\end{pmatrix},\quad X\mapsto\begin{pmatrix}a & -(q-q^{-1})b\\ 0 & b\end{pmatrix},\quad Y\mapsto\begin{pmatrix}b & (q-q^{-1})b\\ 0 & a\end{pmatrix},$$

with $b \neq a$ in order for X and Y to be diagonalizable and with $b \neq q^{\pm 2}a$ to ensure irreducibility. By a change of basis we transform X and Y to a diagonal form:

$$\sigma \mapsto \begin{pmatrix} \frac{(q-q^{-1})b}{b-a} & 1-\frac{(q-q^{-1})^2ab}{(b-a)^2} \\ 1 & -\frac{(q-q^{-1})a}{b-a} \end{pmatrix}, \quad X \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}. \tag{3.3}$$

3. We return to strings of eigenvalues of the Jucys–Murphy elements.

Proposition 1. *Let $\Lambda = (a_1, \dots, a_i, a_{i+1}, \dots, a_n) \in \mathrm{Spec}(J_1, \dots, J_n)$ and let e_Λ be a corresponding vector. Then*

(a) We have $a_i \neq a_{i+1}$.

(b) If $a_{i+1} = q^{\pm 2}a_i$ then $\sigma_i(e_\Lambda) = \pm q^{\pm 1} e_\Lambda$.

(c) If $a_{i+1} \neq q^{\pm 2}a_i$ then $\Lambda' = (a_1, \dots, a_{i+1}, a_i, \dots, a_n) \in \mathrm{Spec}(J_1, \dots, J_n)$; moreover, the vector $\sigma_i(e_\Lambda) - \frac{(q-q^{-1})a_{i+1}}{a_{i+1}-a_i} e_\Lambda$ corresponds to the string Λ' (see (3.3) with $b = a_{i+1}$ and $a = a_i$).

The proof follows directly from the representation theory of the algebra $\hat{H}_2$, described above (cf. the Proposition 4.1 in [28] and the Proposition 3 in [14]).

4. Content strings.

Definition 2. *A content string $(a_1, \dots, a_n)$ is a string of numbers satisfying the following conditions:*

(c1) $a_1 \in \{v_1, \dots, v_m\}$;

(c2) for all $j > 1$: if $a_j = v_k q^{2z}$ for some k and $z \neq 0$ then $\{v_k q^{2(z-1)}, v_k q^{2(z+1)}\} \cap \{a_1, \dots, a_{j-1}\} \neq \varnothing$;

(c3) if $a_i = a_j = v_k q^{2z}$ with $i < j$ for some k and z, then $\{v_k q^{2(z-1)}, v_k q^{2(z+1)}\} \subset \{a_{i+1}, \dots, a_{j-1}\}$.

The set of content strings of length n we denote by $\mathrm{Cont}_m(n)$.

Here is the "cyclotomic" analogue of the Theorem 5.1 in [28] and the Proposition 4 in [14]. We adapt the proof paying attention to places where the restrictions (2.10)–(2.12) are essential.

Proposition 3. *If a string of numbers $(a_1, \dots, a_n)$ belongs to $\mathrm{Spec}(J_1, \dots, J_n)$ then it belongs to $\mathrm{Cont}_m(n)$.*

Proof. Since $J_1 = \tau$ the condition (c1) follows from the characteristic equation for τ.

Assume that (c2) is not true, that is, there is a string $(a_1, \dots, a_n) \in \mathrm{Spec}(J_1, \dots, J_n)$ such that for some $j > 1$, some k and some $z \neq 0$ one has $a_j = v_k q^{2z}$ but $a_i \neq v_k q^{2(z-1)}$ and $a_i \neq v_k q^{2(z+1)}$ for all i smaller than j. By successive applications of the statement (c) of the Proposition 1 we obtain an element of $\mathrm{Spec}(J_1, \dots, J_n)$ with $v_k q^{2z}$ at the first position. The restrictions (2.10)–(2.12) imply $v_k q^{2z} \neq v_i$ for all $i = 1, \dots, m$ and this contradicts the condition (c1).

We prove the condition (c3) by induction on $j-i$. The base of induction is the statement (a) of the Proposition 1. Assume that there are some i and some j such that $i<j$ and $a_i=a_j=v_kq^{2z}$ for some string $(a_1,\dots,a_n)\in \mathrm{Spec}(J_1,\dots,J_n)$. By induction we suppose that the condition (c3) is verified for all i',j' such that $|j'-i'|<j-i$. If $\{v_kq^{2(z-1)},v_kq^{2(z+1)}\}\cap\{a_{i+1},\dots,a_{j-1}\}=\varnothing$ then by an application of the statement (c) of the Proposition 1 we move a_j to the left to the position number $(j-1)$ (note that $(j-1)$ is still greater than i by the statement (a) of the Proposition 1) and obtain an element of $\mathrm{Spec}(J_1,\dots,J_n)$ which contradicts the induction hypothesis. Now assume that only one element from the set $\{v_kq^{2(z-1)},v_kq^{2(z+1)}\}$ is present in $\{a_{i+1},\dots,a_{j-1}\}$. By the induction hypothesis, this element appears only once in $\{a_{i+1},\dots,a_{j-1}\}$. If $j-i>2$ then, by an application of the statement (c) of the Proposition 1, we obtain an element of $\mathrm{Spec}(J_1,\dots,J_n)$ contradicting the induction hypothesis. Thus $j-i=2$ which is impossible because the braid relation $\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}$ is incompatible with assignments $\sigma_i\mapsto \mp q^{\mp1}$ and $\sigma_{i+1}\mapsto \pm q^{\pm1}$ (these values are implied by the statement (b) of the Proposition 1). □

Remark. Let E_S be a vector space with a basis $\{e_\Lambda\}$ whose vectors are labeled by the elements $\Lambda\in\mathrm{Spec}(J_1,\dots,J_n)$. Let E_C be a vector space with a basis $\{e_\mu\}$ whose vectors are labeled by the elements $\mu\in\mathrm{Cont}_m(n)$. By the Proposition 3, $\mathrm{Spec}(J_1,\dots,J_n)\subset\mathrm{Cont}_m(n)$ and so E_S is naturally a vector subspace of E_C. The space E_S is equipped with the action of the algebra $H(m,1,n)$: the operator corresponding to the generator τ is simply J_1; the precise formulas for the action of the elements σ_i, $i=1,\dots,n-1$, are given in the Proposition 1.

The Definition 2 straightforwardly implies that if $(a_1,\dots,a_i,a_{i+1},\dots,a_n)\in\mathrm{Cont}_m(n)$ with $a_{i+1}\neq q^{\pm2}a_i$ then $(a_1,\dots,a_{i+1},a_i,\dots,a_n)\in\mathrm{Cont}_m(n)$. Therefore the operators corresponding to the generators τ and σ_i, $i=1,\dots,n-1$, make sense as the operators in the space E_C. The aim of Subsection 4.3 below is to show that these operators continue to define the action of the algebra $H(m,1,n)$ - now on the, in general, bigger space E_C. One could construct representations working directly with strings but it is illuminating and convenient to interpret the set $\mathrm{Cont}_m(n)$ in the more geometric terms of Young multi-tableaux.

At the end of the whole construction it will then follow, see Section 5, that the spaces E_S and E_C actually coincide.

5. Using "intertwining" operators $U_{i+1}:=\sigma_iJ_i-J_i\sigma_i$, $i=1,\dots,n-1$, it can be proved, as in [14], that:

$$\mathrm{Spec}(J_{i+1})\subset\mathrm{Spec}(J_i)\cup q^{\pm2}\cdot\mathrm{Spec}(J_i). \tag{3.4}$$

Since $\mathrm{Spec}(J_1)\subset\{v_1,\dots,v_m\}$ we arrive at the following conclusion.

Proposition 4. *For all $i=1,\dots,n$,*

$$\mathrm{Spec}(J_i)\subset\{v_kq^{2\,[1-i,i-1]},\ k=1,\dots,m\}\ . \tag{3.5}$$

The Proposition 4 follows also from the Proposition 1 and the Proposition 3. Indeed assume that for a string $(a_1,\dots,a_n)\in\mathrm{Spec}(J_1,\dots,J_n)$ the Proposition 4 does not hold. Let i be the smallest integer for which $a_i\notin\{v_kq^{2\,[1-i,i-1]},\ k=1,\dots,m\}$. Using the statement (c) of the Proposition 1 we move a_i to the left until it reaches the first position in the string and obtain thereby an element of $\mathrm{Spec}(J_1, \dots, J_n)$ which does not verify the condition (c1). This contradicts the Proposition 3.

6. Young m-diagrams and m-tableaux. A Young m-diagram, or m-partition, is an m-tuple of Young diagrams $\lambda^{(m)} = (\lambda_1, \dots, \lambda_m)$. The length of a Young diagram λ is the number of nodes of the diagram and is denoted by $|\lambda|$. By definition the length of an m-tuple $\lambda^{(m)} = (\lambda_1, \dots, \lambda_m)$ is

$$|\lambda^{(m)}| := |\lambda_1| + \cdots + |\lambda_m| \ . \tag{3.6}$$

We recall some standard terminology. For a usual partition λ, a node α is called *removable* if the set of nodes obtained from λ by removing the node α is still a partition; a node β not in λ is called *addable* if the set of nodes obtained from λ by adding β is still a partition.

We extend this terminology for the m-partitions. To this end we define the notion of an *m-node*: an m-node $\alpha^{(m)}$ is a pair (α, p) consisting of a usual node α and an integer p with $1 \le p \le m$. We will refer to α as the *node of the m-node* $\alpha^{(m)}$ and we will write $\mathrm{node}(\alpha^{(m)}) = \alpha$; we will refer to the integer p as the *position of the m-node* $\alpha^{(m)}$ and we will note $\mathrm{pos}(\alpha^{(m)}) = p$. With this definition, an m-partition $\lambda^{(m)}$ is a set of m-nodes such that, for any p between 1 and m, the subset consisting of m-nodes $\alpha^{(m)}$ with $\mathrm{pos}(\alpha^{(m)}) = p$ forms a usual partition.

Let $\lambda^{(m)}$ be an m-partition of length n. An m-node $\alpha^{(m)}$ of $\lambda^{(m)}$ is called *removable* if the set of m-nodes obtained from $\lambda^{(m)}$ by removing $\alpha^{(m)}$ is still an m-partition. An m-node $\beta^{(m)}$ not in $\lambda^{(m)}$ is called *addable* if the set of m-nodes obtained from $\lambda^{(m)}$ by adding $\beta^{(m)}$ is still an m-partition. The m-partition obtained from $\lambda^{(m)}$ by removing any removable m-node $\alpha^{(m)}$ will be denoted by $\lambda^{(m)} \backslash \{\alpha^{(m)}\}$. For any m-partition $\lambda^{(m)}$, we denote by $\mathcal{E}_-(\lambda^{(m)})$ the set of removable m-nodes of $\lambda^{(m)}$ and by $\mathcal{E}_+(\lambda^{(m)})$ the set of addable m-nodes of $\lambda^{(m)}$.

An m-partition whose m-nodes are filled with numbers is called m-tableau.

Let the length of the m-partition $\lambda^{(m)}$ be n. We place now the numbers $1, \dots, n$ in the m-nodes of $\lambda^{(m)}$ in such a way that in every diagram the numbers in the m-nodes are in ascending order along rows and columns in right and down directions. This is a *standard* Young m-tableau of shape $\lambda^{(m)}$.

We associate to each m-node of a Young m-diagram a number (the "content") which is $v_k q^{2(s-r)}$ for the m-node $\alpha^{(m)}$ such that $\mathrm{pos}(\alpha^{(m)}) = k$ and $\mathrm{node}(\alpha^{(m)})$ lies in the line r and column s (equivalently we could say that the m-node $\alpha^{(m)}$ lies in the line r and column s of the k-th diagram of the m-diagram). Note that the notion of content makes sense for an arbitrary m-node of an arbitrary set of m-nodes.

For an arbitrary set of m-nodes, two m-nodes on a same diagonal of the same diagram have the same content which allows us to speak about the "content number" of a diagonal.

Here is an example of a standard Young m-tableau with $m = 2$ and $n = 10$ (the contents of the m-nodes are indicated):

$$\left(\begin{array}{|c|c|c|} \hline {}^{1}\,v_1 & {}^{2}\,v_1q^2 & {}^{4}\,v_1q^4 \\ \hline {}^{6}\,v_1q^{-2} & {}^{9}\,v_1 \\ \cline{1-2} {}^{7}\,v_1q^{-4} \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|} \hline {}^{3}\,v_2 & {}^{8}\,v_2q^2 & {}^{10}\,v_2q^4 \\ \hline {}^{5}\,v_2q^{-2} \\ \cline{1-1} \end{array} \right) . \tag{3.7}$$

Proposition 5. *There is a bijection between the set of standard Young m-tableaux of length n and the set* $\mathrm{Cont}_m(n)$*.*

Proof. To each standard Young m-tableau of length n we associate a string of numbers $(a_1, \ldots, a_n)$ such that a_i for $i = 1, \ldots, n$ is the content of the m-node in which the number i is placed. This string belongs to $\mathrm{Cont}_m(n)$. Indeed the condition (c1) is immediately verified. The condition (c2) is true: if i occupies an m-node with a content $v_k q^{2z}$ with $z \neq 0$ of a standard m-tableau of shape $\lambda^{(m)}$ then there is either an m-node just above in the same column or an m-node just to the left in the same line of the same diagram of $\lambda^{(m)}$; it carries a number less than i and so its content which is $v_k q^{2(z+1)}$ or $v_k q^{2(z-1)}$ appears before $v_k q^{2z}$ in the string. The condition (c3) is true because if $a_i = a_j = v_k q^{2z}$ for some $i < j$ then due to the restrictions (2.10)–(2.12) the m-nodes carrying i and j are on the same diagonal of the same tableau in the m-tableau. Thus the m-node just above the m-node carrying the number j and the m-node just on the left of the m-node carrying the number j are present in $\lambda^{(m)}$; these m-nodes have contents $v_k q^{2(z+1)}$ and $v_k q^{2(z-1)}$ and are occupied by numbers k and l with $k, l \in \{i+1, \ldots, j-1\}$ because the m-tableau is standard.

Conversely to each string $(a_1, \ldots, a_n) \in \mathrm{Cont}_m(n)$ we first associate a set of m-nodes of cardinality n. Additionally, this association distributes the numbers from 1 to n in these m-nodes; each m-node carry one number and different m-nodes carry different numbers. Then we verify that the obtained m-tuple is a standard m-tableau. The construction goes as follows.

The m-nodes are constructed one after another. Given a string $(a_1, \ldots, a_n) \in \mathrm{Cont}_m(n)$ and assuming that $(i-1)$ m-nodes are already constructed, we add at the step number i an m-node on the first non-occupied place of a diagonal whose content number is the value of a_i; we place the number i in this m-node. This construction is unambiguous because the restrictions (2.10)–(2.12) ensure that two different diagonals of such set of m-nodes have different content numbers. The construction of a set of m-nodes of total cardinality n is finished.

We shall now show that the obtained set of m-nodes is a standard m-tableau. Assume by induction that for all $i = 1, \ldots, n-1$ the obtained set of m-nodes is a standard m-tableau after i steps (for $i = 1$ there is nothing to prove; the induction hypothesis is justified because for any $(a_1, \ldots, a_n) \in \mathrm{Cont}_m(n)$, it is clear from the Definition 2 that $(a_1, \ldots, a_i) \in \mathrm{Cont}_m(i)$ for all $i = 1, \ldots, n-1$). It is left to add an m-node in the position dictated by the value $a_n = v_k q^{2z}$, place the number n in it and verify that we obtain a standard m-tableau.

If, for all i, $0 < i < n$, the number a_i is different from the number a_n then the n-th m-node is added at the first place of a corresponding diagonal. If $z = 0$ then there is nothing to prove, so assume that $z > 0$ (the situation with $z < 0$ is similar); the n-th m-node is added in the first line and we have to prove that there is some i, $0 < i < n$, such that $a_i = v_k q^{2(z-1)}$. Suppose that this is not the case; then by condition (c2) of the Definition 2 we have some i, $0 < i < n$, such that $a_i = v_k q^{2(z+1)}$. As $v_k q^{2z}$ is not present in the string before the i^{th} position, the set of m-nodes constructed at the step number i cannot be a standard m-tableau contradicting the induction hypothesis.

Assume that there is some i, $0 < i < n$, such that $a_i = a_n$. We take the largest integer number i satisfying this property. By construction, we add the n-th m-node on the first non-occupied place of the diagonal which contains the m-node carrying the number i. The result is a standard m-tableau only if the m-node just to the right of the m-node carrying the number i and the m-node just below the m-node carrying the number i are present. And this follows from the condition (c3) of the Definition 2 and the induction hypothesis. $\square$

In the example (3.7) the standard Young 2-tableau is associated with the string of numbers:

$$(v_1, v_1q^2, v_2, v_1q^4, v_2q^{-2}, v_1q^{-2}, v_1q^{-4}, v_2q^2, v_1, v_2q^4).$$

Remark. The condition "$z \neq 0$" in the part (c2) of the Definition 2 can be omitted for the Hecke algebras of type A; but this condition is necessary when $m > 1$. It is transparent from the geometric point of view. For the Hecke algebra of type A, if $a_j = 1$ (that is, $z = 0$) for some $j > 1$ then the number j sits on the main diagonal but not in the left upper corner of the standard Young tableau; therefore (both) values q^2 and q^{-2} are present in the string $\{a_1, \ldots, a_{j-1}\}$. However, for $m > 1$, if $a_j = v_k$ for some k and $j > 1$, the number j might occupy a left upper corner of the standard Young m-tableau; in this case it clearly might happen that none of the values $v_k q^2$ and $v_k q^{-2}$ occur in in the string $\{a_1, \ldots, a_{j-1}\}$.

4. Construction of representations

We proceed as in [25]. We first define an algebra structure on the tensor product of the algebra $H(m, 1, n)$ with a free associative algebra generated by the standard m-tableaux corresponding to m-partitions of n; the Baxterized elements are useful here. Then, by evaluation (with the help of the simplest one-dimensional representation of $H(m, 1, n)$) from the right, we build representations.

Using the tensor product of the algebra $H(m, 1, n)$ with the free associative algebra generated by the standard m-tableaux we define and study in Appendix to this Section a structure of a module on the tensor product of two representations corresponding to two m-partitions.

4.1 Baxterized elements

In the definition of the tensor product of the algebra $H(m, 1, n)$ with a free associative algebra we shall frequently use the so-called Baxterized elements.

Define, for any σ_i among the generators $\sigma_1, \ldots, \sigma_{n-1}$ of $H(m, 1, n)$, the Baxterized elements $\sigma_i(\alpha, \beta)$

by

$$\sigma_i(\alpha,\beta) := \sigma_i + (q-q^{-1})\frac{\beta}{\alpha-\beta}\ . \tag{4.1}$$

The parameters α and β are called spectral parameters. We recall some useful relations for the Baxterized generators σ_i. Let

$$f(\alpha,\beta) = \frac{q\alpha - q^{-1}\beta}{\alpha-\beta}\ . \tag{4.2}$$

Proposition 6. *The following relations hold:*

$$\begin{aligned} \sigma_i(\alpha,\beta)\sigma_i(\beta,\alpha) &= f(\alpha,\beta)f(\beta,\alpha), \\ \sigma_i(\alpha,\beta)\sigma_{i+1}(\alpha,\gamma)\sigma_i(\beta,\gamma) &= \sigma_{i+1}(\beta,\gamma)\sigma_i(\alpha,\gamma)\sigma_{i+1}(\alpha,\beta), \\ \sigma_i(\alpha,\beta)\sigma_j(\gamma,\delta) &= \sigma_j(\gamma,\delta)\sigma_i(\alpha,\beta) \qquad\qquad \textit{if}\ \ |i-j|>1. \end{aligned} \tag{4.3}$$

Proof. It is a straightforward and well-known calculation. □

In the construction of representations we shall often verify relations for the Baxterized elements, as in [25]. Relations will be verified for specific values of the spectral parameters. The following Lemma shows that the original relations follow from the relations for the Baxterized elements with fixed values of the spectral parameters.

Lemma 7. *Let A and B be two elements of an arbitrary associative unital algebra $\mathcal{A}$. Denote $A(\alpha,\beta) := A + (q-q^{-1})\frac{\beta}{\alpha-\beta}$ and $B(\alpha,\beta) := B + (q-q^{-1})\frac{\beta}{\alpha-\beta}$ where α and β are parameters.*

(i) If

$$A(\alpha,\beta)A(\beta,\alpha) = f(\alpha,\beta)f(\beta,\alpha)\ ,$$

where f is defined in (4.2), for some (arbitrarily) fixed values of the parameters α and β ($\alpha\neq\beta$) then

$$A^2 - (q-q^{-1})A - 1 = 0\ .$$

(ii) If

$$A^2 - (q-q^{-1})A - 1 = 0\ ,\ \ B^2 - (q-q^{-1})B - 1 = 0$$

and

$$A(\alpha,\beta)B(\alpha,\gamma)A(\beta,\gamma) = B(\beta,\gamma)A(\alpha,\gamma)B(\alpha,\beta)$$

for some (arbitrarily) fixed values of the parameters α,β and γ ($\alpha\neq\beta\neq\gamma\neq\alpha$) then

$$ABA = BAB\ .$$

(iii) If

$$A(\alpha,\beta)B(\gamma,\delta) = B(\gamma,\delta)A(\alpha,\beta)$$

for some (arbitrarily) fixed values of the parameters α,β,γ and δ ($\alpha\neq\beta$ and $\gamma\neq\delta$) then

$$AB = BA\ .$$

Proof. (i) We have

$$
\begin{aligned}
& A(\alpha,\beta)A(\beta,\alpha) = f(\alpha,\beta)f(\beta,\alpha) \\
\Rightarrow\quad & A^2 + (q-q^{-1})\left(\frac{\beta}{\alpha-\beta}+\frac{\alpha}{\beta-\alpha}\right)A - (q-q^{-1})^2\frac{\alpha\beta}{(\beta-\alpha)^2} = \frac{\alpha^2+\beta^2-\alpha\beta(q^2+q^{-2})}{(\beta-\alpha)^2} \\
\Rightarrow\quad & A^2-(q-q^{-1})A-1=0\ .
\end{aligned}
$$

(ii) We have

$$
\begin{aligned}
& A(\alpha,\beta)B(\alpha,\gamma)A(\beta,\gamma) - B(\beta,\gamma)A(\alpha,\gamma)B(\alpha,\beta) = 0 \\
\Rightarrow\quad & ABA - BAB + (q-q^{-1})(A^2-B^2)\frac{\gamma}{\alpha-\gamma} + \\
& +(q-q^{-1})^2(A-B)\left(\frac{\gamma}{\alpha-\gamma}\Big(\frac{\beta}{\alpha-\beta}+\frac{\gamma}{\beta-\gamma}\Big)-\frac{\gamma}{\beta-\gamma}\frac{\beta}{\alpha-\beta}\right)=0 \\
\Rightarrow\quad & ABA - BAB + (q-q^{-1})^2(A-B)\frac{\gamma}{\alpha-\gamma}\left(1+\frac{\beta}{\alpha-\beta}+\frac{\gamma}{\beta-\gamma}-\frac{\beta(\alpha-\gamma)}{(\beta-\gamma)(\alpha-\beta)}\right)=0 \\
\Rightarrow\quad & ABA-BAB=0\ .
\end{aligned}
$$

(iii) It is immediate that $A(\alpha,\beta)B(\gamma,\delta)=B(\gamma,\delta)A(\alpha,\beta)$ implies $AB=BA$. □

4.2 Smash product

We pass to the definition of the associative algebra structure on the product of the algebra $H(m,1,n)$ with a free associative algebra whose generators are indexed by the standard m-tableaux corresponding to m-partitions of n. The resulting algebra we shall denote by $\mathfrak{T}$.

Let $\lambda^{(m)}$ be an m-partition of length n. Consider a set of free generators labeled by standard m-tableaux of shape $\lambda^{(m)}$; for a standard m-tableau $X_{\lambda^{(m)}}$ we denote by $\mathcal{X}_{\lambda^{(m)}}$ the corresponding free generator and by $c(X_{\lambda^{(m)}}|i)$ the content (see the preceding Section for the definition) of the m-node carrying the number i.

In the sequel we shall use the Artin generators of the symmetric group. Recall that the symmetric group (whose group ring is the classical limit of the A-type Hecke algebra H_n) in the Artin presentation is generated by the elements s_i, $i=1,...,n-1$, with the defining relations

$$
\begin{cases}
s_is_{i+1}s_i = s_{i+1}s_is_{i+1} & \text{for all } i=1,\dots,n-2\ , \quad (4.4)\\
s_is_j = s_js_i & \text{for all } i,j=1,\dots,n-1 \text{ such that } |i-j|>1\ , \quad (4.5)\\
s_i^2 = 1 & \text{for all } i=1,\dots,n-1\ . \quad (4.6)
\end{cases}
$$

Let $X_{\lambda^{(m)}}$ be an m-partition of n whose m-nodes are filled with numbers from 1 to n; thus different m-nodes of the m-tableau $X_{\lambda^{(m)}}$ carry different numbers. The m-tableau $X_{\lambda^{(m)}}$ is not necessarily standard. By definition, for such m-tableau $X_{\lambda^{(m)}}$ and any permutation $\pi \in S_n$, the m-tableau $X^{\pi}_{\lambda^{(m)}}$ is obtained from the m-tableau $X_{\lambda^{(m)}}$ by applying the permutation π to the numbers occupying the m-nodes of $X_{\lambda^{(m)}}$; for example $X^{s_i}_{\lambda^{(m)}}$ is the m-tableau obtained from $X_{\lambda^{(m)}}$ by exchanging the positions of the numbers i and $(i+1)$ in the m-tableau $X_{\lambda^{(m)}}$. We thus have by construction:

$$c(X^{\pi}_{\lambda^{(m)}}|i) = c(X_{\lambda^{(m)}}|\pi^{-1}(i)) \tag{4.7}$$

for all $i = 1, \dots, n$.

For a standard m-tableau $X_{\lambda^{(m)}}$, the m-tableau $X^{\pi}_{\lambda^{(m)}}$ is not necessarily standard. As for the generators of the free algebra, we denote the generator corresponding to the m-tableau $X^{\pi}_{\lambda^{(m)}}$ by $\mathcal{X}^{\pi}_{\lambda^{(m)}}$ if the m-tableau $X^{\pi}_{\lambda^{(m)}}$ is standard. And if the m-tableau $X^{\pi}_{\lambda^{(m)}}$ is not standard then we put $\mathcal{X}^{\pi}_{\lambda^{(m)}} = 0$.

Proposition 8. *The relations*

$$\Bigl(\sigma_i + \frac{(q-q^{-1})c(X_{\lambda^{(m)}}|i+1)}{c(X_{\lambda^{(m)}}|i) - c(X_{\lambda^{(m)}}|i+1)}\Bigr) \cdot \mathcal{X}_{\lambda^{(m)}} = \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \Bigl(\sigma_i + \frac{(q-q^{-1})c(X_{\lambda^{(m)}}|i)}{c(X_{\lambda^{(m)}}|i+1) - c(X_{\lambda^{(m)}}|i)}\Bigr) \tag{4.8}$$

and

$$\Bigl(\tau - c(X_{\lambda^{(m)}}|1)\Bigr) \cdot \mathcal{X}_{\lambda^{(m)}} = 0 \tag{4.9}$$

are compatible with the relations for the generators $\tau, \sigma_1, \dots, \sigma_{n-1}$ of the algebra $H(m,1,n)$.

Before the proof we explain the meaning of the word "compatible" in the formulation of the Proposition.

Let $\mathcal{F}$ be the free associative algebra generated by $\tilde{\tau}$, $\tilde{\sigma}_1, \dots, \tilde{\sigma}_{n-1}$. The algebra $H(m,1,n)$ is naturally the quotient of $\mathcal{F}$.

Let $\mathbb{C}[\mathcal{X}]$ be the free associative algebra whose generators $\mathcal{X}_{\lambda^{(m)}}$ range over all standard m-tableaux of shape $\lambda^{(m)}$ for all m-partitions $\lambda^{(m)}$ of n.

Consider an algebra structure on the space $\mathbb{C}[\mathcal{X}] \otimes \mathcal{F}$ for which: (i) the map $\iota_1 : x \mapsto x \otimes 1$, $x \in \mathbb{C}[\mathcal{X}]$, is an isomorphism of $\mathbb{C}[\mathcal{X}]$ with its image with respect to ι_1; (ii) the map $\iota_2 : \phi \mapsto 1 \otimes \phi$, $\phi \in \mathcal{F}$, is an isomorphism of $\mathcal{F}$ with its image with respect to ι_2; (iii) the formulas (4.8)-(4.9), extended by associativity, provide the rules to rewrite elements of the form $(1 \otimes \phi)(x \otimes 1)$, $x \in \mathbb{C}[\mathcal{X}]$, $\phi \in \mathcal{F}$, as elements of $\mathbb{C}[\mathcal{X}] \otimes \mathcal{F}$.

The "compatibility" means that we have an induced structure of an associative algebra, denoted by $\mathfrak{T}$, on the space $\mathbb{C}[\mathcal{X}] \otimes H(m,1,n)$. More precisely, if we multiply any relation of the cyclotomic Hecke algebra $H(m,1,n)$ (the relation is viewed as an element of the free algebra $\mathcal{F}$) from the right by a generator $\mathcal{X}_{\lambda^{(m)}}$ (this is a combination of the form "a relation of $H(m,1,n)$ times $\mathcal{X}_{\lambda^{(m)}}$") and use the "instructions" (4.8)-(4.9) to move all appearing $\mathcal{X}$'s to the left (the free generator changes but the expression stays always linear in $\mathcal{X}$) then we obtain a linear combination of terms of the form "$\mathcal{X}^{\pi}_{\lambda^{(m)}}$, $\pi \in S_n$, times a relation of $H(m,1,n)$".

Proof. We rewrite the relation (4.8) using the Baxterized form of the elements σ_i:

$$\sigma_i\Bigl(c(X_{\lambda^{(m)}}|i), c(X_{\lambda^{(m)}}|i+1)\Bigr) \cdot \mathcal{X}_{\lambda^{(m)}} = \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \sigma_i\Bigl(c(X_{\lambda^{(m)}}|i+1), c(X_{\lambda^{(m)}}|i)\Bigr)\ .$$

For brevity we denote $c^{(k)} := c(X_{\lambda^{(m)}}|k)$ for all $k = 1, \dots, n$.

We shall check the compatibility of the "instructions" (4.8)-(4.9) with the set of defining relations (2.1)–(2.4) and (2.7)–(2.8). We start with the relations involving the generators σ_i only. Here we use the Baxterized form of the relations and the Lemma 7.

Below we shall use without mentioning the inequalities $c^{(k)} \neq c^{(k+1)}$, $c^{(k)} \neq c^{(k+2)}$, $c^{(k+1)} \neq c^{(k+2)}$ (for all k) which follow from the restrictions (2.10)–(2.12).

(a) We consider first the relation $\sigma_i^2 = (q - q^{-1})\sigma_i + 1$. If the m-tableau $X^{s_i}_{\lambda^{(m)}}$ is standard then we analyze this relation in its equivalent form, given in (i) in the Lemma 7. We have

$$\begin{array}{rl} & \sigma_i(c^{(i+1)}, c^{(i)})\ \sigma_i(c^{(i)}, c^{(i+1)}) \cdot \mathcal{X}_{\lambda^{(m)}} \\ = & \sigma_i(c^{(i+1)}, c^{(i)}) \cdot \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i+1)}, c^{(i)}) \\ = & \mathcal{X}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i)}, c^{(i+1)})\ \sigma_i(c^{(i+1)}, c^{(i)})\ . \end{array} \tag{4.10}$$

We used (4.7) in the second equality. Therefore,

$$\begin{array}{rl} & \Big(\sigma_i(c^{(i+1)}, c^{(i)})\ \sigma_i(c^{(i)}, c^{(i+1)}) - f(c^{(i+1)}, c^{(i)}) f(c^{(i)}, c^{(i+1)})\Big) \cdot \mathcal{X}_{\lambda^{(m)}} \\ = & \mathcal{X}_{\lambda^{(m)}} \cdot \Big(\sigma_i(c^{(i)}, c^{(i+1)})\ \sigma_i(c^{(i+1)}, c^{(i)}) - f(c^{(i+1)}, c^{(i)}) f(c^{(i)}, c^{(i+1)})\Big) \end{array}$$

and the compatibility for this relation is verified since the expression on the right of $\mathcal{X}_{\lambda^{(m)}}$ belongs, by the Proposition 6, to the ideal generated by relations.

If the m-tableau $X^{s_i}_{\lambda^{(m)}}$ is not standard then $(i+1)$ sits in the same tableau of the m-tableau $X_{\lambda^{(m)}}$ as i and is situated directly to the right or directly down with respect to i. In this situation the relation (4.8) can be rewritten as $\sigma_i \mathcal{X}_{\lambda^{(m)}} = w \mathcal{X}_{\lambda^{(m)}}$, where w is equal to either q or $(-q^{-1})$, and the verification of the compatibility of the relation $\sigma_i^2 - (q - q^{-1})\sigma_i - 1 = 0$ with the instructions (4.8)-(4.9) is straightforward.

(b) The relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ we analyze in its equivalent form, given in (ii) in the Lemma 7. We have

$$\begin{array}{rl} & \sigma_i(c^{(i+1)}, c^{(i+2)}) \sigma_{i+1}(c^{(i)}, c^{(i+2)}) \sigma_i(c^{(i)}, c^{(i+1)}) \cdot \mathcal{X}_{\lambda^{(m)}} \\ = & \mathcal{X}^{s_i s_{i+1} s_i}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i+2)}, c^{(i+1)}) \sigma_{i+1}(c^{(i+2)}, c^{(i)}) \sigma_i(c^{(i+1)}, c^{(i)}) \end{array} \tag{4.11}$$

and

$$\begin{array}{rl} & \sigma_{i+1}(c^{(i)}, c^{(i+1)}) \sigma_i(c^{(i)}, c^{(i+2)}) \sigma_{i+1}(c^{(i+1)}, c^{(i+2)}) \cdot \mathcal{X}_{\lambda^{(m)}} \\ = & \mathcal{X}^{s_{i+1} s_i s_{i+1}}_{\lambda^{(m)}} \cdot (\sigma_{i+1}(c^{(i+1)}, c^{(i)}) \sigma_i(c^{(i+2)}, c^{(i)}) \sigma_{i+1}(c^{(i+2)}, c^{(i+1)})\ . \end{array} \tag{4.12}$$

We used several times the relations (4.7).

One might think that, as for the relation $\sigma_i^2 - (q - q^{-1})\sigma_i - 1 = 0$, we should consider separately the situations when in the process of transformation the m-tableau becomes non-standard. However one verifies that for an arbitrary standard m-tableau $Y_{\lambda^{(m)}}$:

- if the m-tableau $Y^{s_i}_{\lambda^{(m)}}$ is not standard then the m-tableaux $Y^{s_is_{i+1}}_{\lambda^{(m)}}$ and $Y^{s_is_{i+1}s_i}_{\lambda^{(m)}}$ are not standard as well;
- if the m-tableau $Y^{s_{i+1}}_{\lambda^{(m)}}$ is not standard then the m-tableaux $Y^{s_{i+1}s_i}_{\lambda^{(m)}}$ and $Y^{s_{i+1}s_is_{i+1}}_{\lambda^{(m)}}$ are not standard as well.

It then follows that

- if the m-tableau $Y^{s_is_{i+1}s_i}_{\lambda^{(m)}}$ is standard then the m-tableaux $Y^{s_i}_{\lambda^{(m)}}$ and $Y^{s_is_{i+1}}_{\lambda^{(m)}}$ are standard as well;
- if the m-tableau $Y^{s_{i+1}s_is_{i+1}}_{\lambda^{(m)}}$ is standard then the m-tableaux $Y^{s_{i+1}}_{\lambda^{(m)}}$ and $Y^{s_{i+1}s_i}_{\lambda^{(m)}}$ are standard as well.

Therefore, we cannot return to a standard m-tableau if an intermediate m-tableau was not standard. Thus the equalities (4.11) and (4.12) are always valid, in contrast to (4.10).

We replace $\mathcal{X}^{s_{i+1}s_is_{i+1}}_{\lambda^{(m)}}$ by $\mathcal{X}^{s_is_{i+1}s_i}_{\lambda^{(m)}}$ in the right hand side of (4.12) and subtract (4.12) from (4.11). In the result, the expression on the right of $\mathcal{X}^{s_is_{i+1}s_i}_{\lambda^{(m)}}$ belongs, by the Proposition 6, to the ideal generated by relations.

(c) The relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j|>1$ we analyze in its equivalent form, given in (iii) in the Lemma 7. We have

$$\begin{aligned} & \Bigl(\sigma_i(c^{(i)},c^{(i+1)})\sigma_j(c^{(j)},c^{(j+1)}) - \sigma_j(c^{(j)},c^{(j+1)})\sigma_i(c^{(i)},c^{(i+1)})\Bigr)\cdot \mathcal{X}_{\lambda^{(m)}} \\ = \ & \mathcal{X}^{s_is_j}_{\lambda^{(m)}}\cdot\Bigl(\sigma_i(c^{(i+1)},c^{(i)})\sigma_j(c^{(j+1)},c^{(j)}) - \sigma_j(c^{(j+1)},c^{(j)})\sigma_i(c^{(i+1)},c^{(i)})\Bigr)\ . \end{aligned} \tag{4.13}$$

Again, as for the previous relation, a direct inspection shows that (4.13) is always valid.

The expression on the right of $\mathcal{X}^{s_is_j}_{\lambda^{(m)}}$ in the right hand side of (4.13) belongs again to the ideal generated by relations by the Proposition 6.

(d) It is left to analyze the relations including the generator τ.

The verification of the compatibility of the relations $(\tau-v_1)\dots(\tau-v_m)=0$ and $\tau\sigma_i=\sigma_i\tau$ for $i>1$ with the instructions (4.8)-(4.9) is immediate.

The compatibility of the remaining relation $\tau\sigma_1\tau\sigma_1=\sigma_1\tau\sigma_1\tau$ with the instructions (4.8)-(4.9) is a direct consequence of the Lemma below. □

Lemma 9. *The relations (4.8)-(4.9) imply the relations:*

$$\bigl(J_i - c(X_{\lambda^{(m)}}|i)\bigr)\cdot\mathcal{X}_{\lambda^{(m)}} = 0 \quad \textit{for all } i=1,\dots,n\ . \tag{4.14}$$

Proof. For brevity we denote $c^{(k)} := c(X_{\lambda^{(m)}}|k)$ for all $k=1,\dots,n$.

Recall that $J_1=\tau$ and $J_{i+1}=\sigma_iJ_i\sigma_i$. We use induction on i. For $i=1$ the relation (4.14) is the relation (4.9).

Assume first that the m-tableau $X^{s_i}_{\lambda^{(m)}}$ is standard. Then

$$\begin{aligned}
\sigma_i J_i \sigma_i \cdot \mathcal{X}_{\lambda^{(m)}} &= \sigma_i J_i \cdot \Big(-(q-q^{-1}) \tfrac{c^{(i+1)}}{c^{(i)}-c^{(i+1)}} \mathcal{X}_{\lambda^{(m)}} + \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i+1)}, c^{(i)}) \Big) \\
&= \sigma_i \cdot \Big(-(q-q^{-1}) c^{(i)} \tfrac{c^{(i+1)}}{c^{(i)}-c^{(i+1)}} \mathcal{X}_{\lambda^{(m)}} + c^{(i+1)} \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i+1)}, c^{(i)}) \Big) \\
&= (q-q^{-1})^2 c^{(i)} \tfrac{c^{(i+1)} c^{(i+1)}}{(c^{(i+1)}-c^{(i)})^2} \mathcal{X}_{\lambda^{(m)}} - (q-q^{-1}) \tfrac{c^{(i)} c^{(i+1)}}{c^{(i)}-c^{(i+1)}} \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i+1)}, c^{(i)}) \\
&\quad -(q-q^{-1}) \tfrac{c^{(i+1)} c^{(i)}}{c^{(i+1)}-c^{(i)}} \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i+1)}, c^{(i)}) + c^{(i+1)} \mathcal{X}_{\lambda^{(m)}} \cdot \sigma_i(c^{(i)}, c^{(i+1)}) \sigma_i(c^{(i+1)}, c^{(i)}) \\
&= c^{(i+1)} \Big((q-q^{-1})^2 \tfrac{c^{(i)} c^{(i+1)}}{(c^{(i+1)}-c^{(i)})^2} + \tfrac{c^{(i)} c^{(i)} + c^{(i+1)} c^{(i+1)} - c^{(i)} c^{(i+1)} (q^2+q^{-2})}{(c^{(i+1)}-c^{(i)})^2} \Big) \mathcal{X}_{\lambda^{(m)}} \\
&= c^{(i+1)} \mathcal{X}_{\lambda^{(m)}} \ .
\end{aligned}$$

Here we moved the elements σ_i to the right, using the relation (4.8); we used then relations (4.7), the induction hypothesis and the first relation in the Proposition 6.

Then assume that the m-tableau $X^{s_i}_{\lambda^{(m)}}$ is not standard. It means that the m-nodes carrying numbers i and $(i+1)$ are adjacent (neighbors in a same line or a same column of a tableau in the m-tableau). In this situation we have $c^{(i)} = q^{\pm 2} c^{(i+1)}$ and

$$\begin{aligned}
\sigma_i J_i \sigma_i \cdot \mathcal{X}_{\lambda^{(m)}} &= \sigma_i J_i \cdot \Big(-(q-q^{-1}) \tfrac{c^{(i+1)}}{q^{\pm 2} c^{(i+1)} - c^{(i+1)}} \Big) \mathcal{X}_{\lambda^{(m)}} \\
&= \sigma_i \cdot \Big(-(q-q^{-1}) \tfrac{q^{\pm 2} c^{(i+1)} c^{(i+1)}}{q^{\pm 2} c^{(i+1)} - c^{(i+1)}} \Big) \mathcal{X}_{\lambda^{(m)}} \\
&= (q-q^{-1})^2 \ \tfrac{q^{\pm 2} c^{(i+1)} c^{(i+1)} c^{(i+1)}}{(q^{\pm 2} c^{(i+1)} - c^{(i+1)})^2} \mathcal{X}_{\lambda^{(m)}} \\
&= c^{(i+1)} \mathcal{X}_{\lambda^{(m)}} \ .
\end{aligned}$$

Here we moved the elements σ_i to the right using the relation (4.8); we used then the induction hypothesis. □

4.3 Representations

The Proposition 8 provides an effective tool for the construction of representations of the cyclotomic Hecke algebra $H(m, 1, n)$.

Let $|\rangle$ be a "vacuum" - a basic vector of a one-dimensional $H(m, 1, n)$-module; for example,

$$\sigma_i |\rangle = q |\rangle \ \text{ for all } i \text{ and } \ \tau |\rangle = v_1 |\rangle \ . \tag{4.15}$$

Moving, in the expressions $\phi \mathcal{X}_{\lambda^{(m)}} |\rangle$, $\phi \in H(m, 1, n)$, the elements $\mathcal{X}$'s to the left and using the module structure (4.15), we build, due to the compatibility, a representation of $H(m, 1, n)$ on the vector space $U_{\lambda^{(m)}}$ with the basis $\mathcal{X}_{\lambda^{(m)}} |\rangle$. We shall, by a slight abuse of notation, denote the symbol $\mathcal{X}_{\lambda^{(m)}} |\rangle$ again by $\mathcal{X}_{\lambda^{(m)}}$. This procedure leads to the following formulas for the action of the generators $\tau, \sigma_1, \dots, \sigma_{n-1}$ on the basis vectors $\mathcal{X}_{\lambda^{(m)}}$ of $U_{\lambda^{(m)}}$:

$$\sigma_i \,:\, \mathcal{X}_{\lambda^{(m)}} \mapsto \ \frac{(q-q^{-1}) c(X_{\lambda^{(m)}} | i+1)}{c(X_{\lambda^{(m)}} | i+1) - c(X_{\lambda^{(m)}} | i)} \mathcal{X}_{\lambda^{(m)}} + \ \frac{q c(X_{\lambda^{(m)}} | i+1) - q^{-1} c(X_{\lambda^{(m)}} | i)}{c(X_{\lambda^{(m)}} | i+1) - c(X_{\lambda^{(m)}} | i)} \mathcal{X}^{s_i}_{\lambda^{(m)}} \tag{4.16}$$

and

$$\tau\ :\ \mathcal{X}_{\lambda^{(m)}} \mapsto c(X_{\lambda^{(m)}}|1)\mathcal{X}_{\lambda^{(m)}}\ . \tag{4.17}$$

As before, it is assumed here that $\mathcal{X}^{s_i}_{\lambda^{(m)}} = 0$ if $X^{s_i}_{\lambda^{(m)}}$ is not a standard m-tableau. Denote this $H(m,1,n)$-module by $V_{\lambda^{(m)}}$.

Assume that the m-tableau $X^{s_i}_{\lambda^{(m)}}$ is standard. The two-dimensional subspace of $U_{\lambda^{(m)}}$ with the basis $\{\mathcal{X}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}}\}$ is σ_i-invariant. For the future convenience we write down the matrix giving the action of the generator σ_i on this two-dimensional subspace:

$$\frac{1}{c^{(i+1)}-c^{(i)}}\begin{pmatrix} (q-q^{-1})c^{(i+1)} & q^{-1}c^{(i+1)}-qc^{(i)} \\ qc^{(i+1)}-q^{-1}c^{(i)} & -(q-q^{-1})c^{(i)} \end{pmatrix}\ , \tag{4.18}$$

where we denoted $c^{(i)} = c(X_{\lambda^{(m)}}|i)$ and $c^{(i+1)} = c(X_{\lambda^{(m)}}|i+1)$.

Remarks.

(a) In our construction of representations, the Baxterization of the generator τ (which can be found in [16]) is never used while the Baxterized generators σ_i, $i=1,\dots,n-1$, appear. The relation (4.9) says that τ, placed before $\mathcal{X}_{\lambda^{(m)}}$, can be immediately replaced by a number. This is similar to the situation with the element σ_1 in the representation theory of the Hecke ($m=1$) algebras. Indeed if $m=1$ then for any standard tableau X_λ, the tableau $X_\lambda s_1$ is non-standard and so σ_1, placed before $\mathcal{X}_\lambda$, can be immediately replaced by a number; in particular, the action of σ_1 is given by a diagonal matrix in the basis $\mathcal{X}_\lambda|\rangle$ of U_λ; the behavior of τ extends this phenomenon to $m>1$.

(b) It follows from the preceding remark that the action of the generators in the constructed representations do not depend on the value of τ on the vacuum $|\rangle$. Moreover the constructed representations do not depend (up to isomorphism) on the value of σ_i, $i=1,\dots,n-1$, on the vacuum. Indeed if we take for the vacuum a basic vector $|\rangle'$ of the one-dimensional $H(m,1,n)$-module such that $\sigma_i|\rangle' = -q^{-1}|\rangle'$ for all i and $\tau|\rangle' = v_1|\rangle'$, the procedure described in this subsection leads to representations $\tilde{V}_{\lambda^{(m)}}$ of $H(m,1,n)$. By construction, $V_{\lambda^{(m)}}$ and $\tilde{V}_{\lambda^{(m)}}$ have the same underlying vector space $U_{\lambda^{(m)}}$. It is straightforward to check that for any m-partition $\lambda^{(m)}$ there is an isomorphism between $H(m,1,n)$-modules $V_{\lambda^{(m)}}$ and $\tilde{V}_{\lambda^{(m)}}$; the operators for the representation $\tilde{V}_{\lambda^{(m)}}$ are obtained from the operators for the representation $V_{\lambda^{(m)}}$ by the following diagonal change of basis in $U_{\lambda^{(m)}}$:

$$\mathcal{X}_{\lambda^{(m)}} \mapsto \mathfrak{c}_{\mathcal{X}_{\lambda^{(m)}}}\mathcal{X}_{\lambda^{(m)}}\ ,\quad \text{where}\quad \mathfrak{c}_{\mathcal{X}_{\lambda^{(m)}}} = \prod_{i\,:\,\mathcal{X}^{s_i}_{\lambda^{(m)}}\neq 0} \big(q\ c(X_{\lambda^{(m)}}|i) - q^{-1}c(X_{\lambda^{(m)}}|i+1)\big)\ .$$

By construction, $\mathfrak{c}_{\mathcal{X}_{\lambda^{(m)}}} \neq 0$.

(c) In the Hecke situation ($m=1$) the coefficients appearing in the action of the generators can be expressed in terms of the lengths $l_{j,j+1}$ between nodes (see, e.g., [25]). We do not define the length between nodes which do not belong to the same tableau of the m-tableau; the form, referring to lengths, of the action is not useful any more.

(d) The constructed action of the generators in the representations $V_{\lambda^{(m)}}$ coincides with the action given in [1].

(e) The action of the intertwining operators $U_{i+1} = \sigma_i J_i - J_i \sigma_i$, $i = 1, \dots, n-1$, (see paragraph **5** of Section 3) in a representation $V_{\lambda^{(m)}}$ is:

$$U_{i+1}(\mathcal{X}_{\lambda^{(m)}}) = \left(q^{-1}c^{(i)} - qc^{(i+1)}\right) \mathcal{X}^{s_i}_{\lambda^{(m)}} \ , \tag{4.19}$$

where $c^{(i)} = c(X_{\lambda^{(m)}}|i)$, $i = 1, \dots, n$. Indeed we rewrite $U_{i+1} = \sigma_i J_i - \sigma_i^{-1} J_{i+1} = \sigma_i (J_i - J_{i+1}) + (q - q^{-1}) J_{i+1}$, so, by the Lemma 9,

$$U_{i+1}(\mathcal{X}_{\lambda^{(m)}}) = (c^{(i)} - c^{(i+1)}) \left(\sigma_i(\mathcal{X}_{\lambda^{(m)}}) + \frac{(q - q^{-1})c^{(i+1)}}{c^{(i)} - c^{(i+1)}} \mathcal{X}_{\lambda^{(m)}} \right) \ .$$

Using (4.16) we obtain the formula (4.19).

4.4 Scalar product

1. The representations constructed on the spaces $U_{\lambda^{(m)}}$, where $\lambda^{(m)}$ is an m-partition of length n, are analogues for $H(m,1,n)$ of the seminormal representations of the symmetric group. We compute here analogues, for the representations spaces of $H(m,1,n)$, of invariant scalar products on representation spaces for the symmetric group.

Let

$$\mathfrak{D} := \{1 + q^2 + \dots + q^{2N}\}_{N=1,\dots,n} \cup \{q^{2i} v_j - v_k\}_{i,j,k:\ j \neq k,\ -n<i<n}$$

and let $\mathfrak{R}$ be the ring $\mathbb{C}[q, q^{-1}, v_1, v_1^{-1}, \dots, v_m, v_m^{-1}]$ of Laurent polynomials in variables $q, v_1, \dots, v_m$ localized with respect to the multiplicative set generated by $\mathfrak{D}$. Denote by ω the involution on $\mathbb{C}[q, q^{-1}, v_1, v_1^{-1}, \dots, v_m, v_m^{-1}]$ which sends q to q^{-1} and v_j to v_j^{-1}, $j = 1, \dots, m$. The involution ω is compatible with the localization and thus extends to $\mathfrak{R}$. Let U be a free module over $\mathfrak{R}$. We shall call *ω-sesquilinear scalar product* a linear map $\langle , \rangle \colon U \otimes_{\mathbb{C}} U \to \mathfrak{R}$ of complex vector spaces with the property

$$\langle f\, u, g\, v \rangle = f\, \omega(g)\, \langle u, v \rangle \ , \ \ f, g \in \mathfrak{R} \ .$$

We shall work here with the generic cyclotomic Hecke algebra, that is, the cyclotomic Hecke algebra over the ring $\mathfrak{R}$. We denote by the same symbol $U_{\lambda^{(m)}}$ the vector space, now over the ring $\mathfrak{R}$, with the basis $\{\mathcal{X}_{\lambda^{(m)}}\}$.

Let $\lambda^{(m)}$ be an m-partition and let $X_{\lambda^{(m)}}$ and $X'_{\lambda^{(m)}}$ be two different standard m-tableaux of shape $\lambda^{(m)}$. For brevity we set $c^{(i)} = c(X_{\lambda^{(m)}}|i)$ for all $i = 1, \dots, n$. We introduce an ω-sesquilinear scalar product on $U_{\lambda^{(m)}}$ defined on the basis $\{\mathcal{X}_{\lambda^{(m)}}\}$ by

$$\langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}'_{\lambda^{(m)}} \rangle = 0 \ , \tag{4.20}$$

$$\langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle = \prod_{j,k:\ j<k\,,\ c^{(j)} \neq c^{(k)}\,,\ c^{(j)} \neq c^{(k)} q^{\pm 2}} \frac{q^{-1}c^{(j)} - qc^{(k)}}{c^{(j)} - c^{(k)}} \ . \tag{4.21}$$

Notice that, if $X^{s_i}_{\lambda^{(m)}}$ is a standard m-tableau, we have

$$\langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}} \rangle = \frac{qc^{(i)} - q^{-1}c^{(i+1)}}{q^{-1}c^{(i)} - qc^{(i+1)}} \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle \ . \tag{4.22}$$

As before, the generators $\tau, \sigma_1, \dots, \sigma_{n-1}$ of the algebra $H(m,1,n)$ act on the space $U_{\lambda^{(m)}}$ according to the formulas (4.16)–(4.17). We shall show that the ω-sesquilinear scalar product (4.20)-(4.21) is $H(m,1,n)$-invariant in the sense that

$$\langle \mathsf{a}(u), \mathsf{a}(v) \rangle = \langle u, v \rangle \quad \text{for} \quad \mathsf{a} = \tau, \sigma_1, \dots, \sigma_{n-1} \ . \tag{4.23}$$

It is immediate that (4.20)–(4.21) are invariant under the action of the generator τ of $H(m,1,n)$. The verification of the invariance of (4.20) under the action of the generator σ_i of $H(m,1,n)$ is non-trivial only if $X^{s_i}_{\lambda^{(m)}}$ is standard and $X'_{\lambda^{(m)}} = X^{s_i}_{\lambda^{(m)}}$. This verification is done by a straightforward calculation of $\langle \sigma_i(\mathcal{X}_{\lambda^{(m)}}), \sigma_i(\mathcal{X}^{s_i}_{\lambda^{(m)}}) \rangle$; it is equal to

$$-\frac{(q-q^{-1})c^{(i+1)}}{(c^{(i)} - c^{(i+1)})^2} \Big((qc^{(i)} - q^{-1}c^{(i+1)}) \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle + (qc^{(i+1)} - q^{-1}c^{(i)}) \langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}} \rangle \Big) \ ,$$

which is equal to 0 due to (4.22).

If $X^{s_i}_{\lambda^{(m)}}$ is not standard then $\sigma_i(\mathcal{X}_{\lambda^{(m)}}) = \epsilon q^{\epsilon} \mathcal{X}_{\lambda^{(m)}}$, $\epsilon = \pm 1$, and the invariance of (4.21) under the action of σ_i follows.

If $X^{s_i}_{\lambda^{(m)}}$ is standard, then a straightforward calculation gives that $\langle \sigma_i(\mathcal{X}_{\lambda^{(m)}}), \sigma_i(\mathcal{X}_{\lambda^{(m)}}) \rangle$ is equal to

$$\frac{(q-q^{-1})^2 c^{(i)} c^{(i+1)} \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle + (q^{-1}c^{(i)} - qc^{(i+1)})^2 \langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}} \rangle}{(c^{(i)} - c^{(i+1)})^2} \ .$$

Using (4.22) one obtains

$$\langle \sigma_i(\mathcal{X}_{\lambda^{(m)}}), \sigma_i(\mathcal{X}_{\lambda^{(m)}}) \rangle = \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle \ .$$

The proof of the $H(m,1,n)$-invariance of (4.20)–(4.21) is finished. As a direct consequence, (4.20)–(4.21) are invariant under the action of any product of the generators of the algebra $H(m,1,n)$.

2. In the generic situation, analogues for $H(m,1,n)$ of the orthogonal representations of the symmetric group are defined over an extension $\tilde{\mathfrak{R}}$ of the ring $\mathfrak{R}$. Let $\mathfrak{F}_{\lambda^{(m)}}$ be the set of all standard m-tableaux of shape $\lambda^{(m)}$. Let also $\mathfrak{E}_{X_{\lambda^{(m)}}} := \{(j,k) | j < k \, , \, c^{(j)} \neq c^{(k)} \, , \, c^{(j)} \neq c^{(k)} q^{\pm 2}\}$ (this is the set over which the product in the right hand side of (4.21) is taken). Introduce, for each standard m-tableau $X_{\lambda^{(m)}}$, a collection of variables $\varsigma^{jk}_{X_{\lambda^{(m)}}}$ and let

$$\tilde{\mathfrak{R}} := \mathfrak{R} \left[\varsigma^{jk}_{X_{\lambda^{(m)}}} \right]_{X_{\lambda^{(m)}} \in \mathfrak{F}_{\lambda^{(m)}}, (j,k) \in \mathfrak{E}_{X_{\lambda^{(m)}}}} \Big/ \mathfrak{I} \ ,$$

where $\mathfrak{I}$ is the ideal generated by $\left\{ \left(\varsigma^{jk}_{X_{\lambda(m)}} \right)^2 - \frac{q^{-1}c^{(j)} - qc^{(k)}}{c^{(j)} - c^{(k)}} \right\}_{X_{\lambda(m)} \in \mathfrak{F}_{\lambda(m)}, (j,k) \in \mathfrak{E}_{X_{\lambda(m)}}}$. Since each factor $\frac{q^{-1}c^{(j)} - qc^{(k)}}{c^{(j)} - c^{(k)}}$ in the product in the right hand side of (4.21) is stable with respect to the involution ω, one can extend the involution ω to the ring $\tilde{\mathfrak{R}}$ by the rule $\omega \left(\varsigma^{jk}_{X_{\lambda(m)}} \right) = \varsigma^{jk}_{X_{\lambda(m)}}$ for all $X_{\lambda(m)} \in \mathfrak{F}_{\lambda(m)}$ and $(j,k) \in \mathfrak{E}_{X_{\lambda(m)}}$.

An analogue for $H(m,1,n)$ of the orthogonal representation of the symmetric group is obtained by performing the following diagonal change of basis in the $\tilde{\mathfrak{R}}$-module $\tilde{\mathfrak{R}} \otimes_{\mathfrak{R}} U_{\lambda(m)}$:

$$\mathcal{X}_{\lambda(m)} \mapsto \tilde{\mathcal{X}}_{\lambda(m)} := \mathfrak{d}_{\mathcal{X}_{\lambda(m)}} \mathcal{X}_{\lambda(m)} \ , \quad \text{where} \ \ \mathfrak{d}_{\mathcal{X}_{\lambda(m)}} = \prod_{j,k\colon (j,k) \in \mathfrak{E}_{X_{\lambda(m)}}} \varsigma^{jk}_{X_{\lambda(m)}} \tag{4.24}$$

for any standard m-tableau $X_{\lambda(m)}$ of shape $\lambda^{(m)}$. In the new basis $\{\tilde{\mathcal{X}}_{\lambda(m)}\}$, the operators for the generators of $H(m,1,n)$ are orthogonal with respect to the ω-sesquilinear form. Namely let A be the matrix of the generator $\mathsf{a} \in \{\tau, \sigma_1, \ldots, \sigma_{n-1}\}$ of $H(m,1,n)$ in the basis $\{\tilde{\mathcal{X}}_{\lambda(m)}\}$ then

$$\mathsf{A}\, \omega(\mathsf{A})^T = \mathbf{Id} \ , \tag{4.25}$$

where $\mathbf{Id}$ is the identity matrix and, given a matrix x, x^T means the transposed matrix .

Note that if q and v_j, $j = 1, \ldots, m$, are numerical parameters whose values are of norm 1 then the ω-sesquilinear form is just the usual Hermitian form and the matrices of the generators of $H(m,1,n)$ are unitary in the usual sense.

3. Reintroduce the deformation parameters $q, v_1, \ldots, v_m$ in the notation for the cyclotomic Hecke algebra: $H_{q,v_1,\ldots,v_m}(m,1,n)$. There is another way to interpret the formulas (4.20)-(4.21). Namely these formulas define a pairing between the representation spaces $U_{\lambda(m)}$ and $U'_{\lambda(m)}$ where the first space $U_{\lambda(m)}$ carries the representation of the algebra $H_{q,v_1,\ldots,v_m}(m,1,n)$ and the second space $U'_{\lambda(m)}$ carries the representation of the algebra $H_{q^{-1},v_1^{-1},\ldots,v_m^{-1}}(m,1,n)$. The vector spaces $U_{\lambda(m)}$ and $U'_{\lambda(m)}$ are naturally isomorphic. The formula (4.23), stating the invariance of the pairing, holds; now in the formula (4.23), $x(u)$ is to be understood as the result of the action of the generator $x \in H_{q,v_1,\ldots,v_m}(m,1,n)$ on the vector $u \in U_{\lambda(m)}$ while $x(v)$ is the result of the action of the generator $x \in H_{q^{-1},v_1^{-1},\ldots,v_m^{-1}}(m,1,n)$ on the vector $v \in U'_{\lambda(m)}$. The pairing is bilinear in the usual sense, $\langle fu, gv \rangle = fg\langle u,v \rangle$, $f,g \in \mathfrak{R}$.

4. For $m = 1$, that is, for the usual Hecke algebra $H(1,1,n)$, we have $c(i) = q^{2cc(i)}$ where $cc(i)$ is the classical content of the node occupied by i and the formula (4.21) can be rewritten in the following form

$$\langle \mathcal{X}_{\lambda(m)}, \mathcal{X}_{\lambda(m)} \rangle = \prod_{j,k\colon j<k\, ,\, c^{(j)} \neq c^{(k)}\, ,\, c^{(j)} \neq c^{(k)} q^{\pm 2}} \frac{(cc(j) - cc(k) - 1)_q}{(cc(j) - cc(k))_q} \ . \tag{4.26}$$

5. Let $\rho\colon \hat{H}_n \to \mathrm{End}(\mathcal{V})$ be an irreducible representation of the affine Hecke algebra $\hat{H}_n$ on a complex vector space $\mathcal{V}$ of finite dimension L. Assume that the operator $\rho(\tau)$ is diagonalizable and the spectrum of $\rho(\tau)$ is $\{(v_1)_{\mathfrak{l}_1}, (v_2)_{\mathfrak{l}_2}, \dots, (v_m)_{\mathfrak{l}_m}\}$; here the numbers $\{v_1, v_2, \dots, v_m\}$ are pairwise different, $\mathfrak{l}_j$ is the multiplicity of the eigenvalue v_j, $j = 1, \dots, m$. Then the representation ρ passes through the cyclotomic quotient $H_{q,v_1,\dots,v_m}(m,1,n)$ of the affine Hecke algebra. Assume that the parameters $q, v_1, \dots, v_m$ satisfy the restrictions (2.10)-(2.12). By the completeness result from Section 5, the representation ρ is isomorphic, as the representation of $H_{q,v_1,\dots,v_m}(m,1,n)$, to $V_{\lambda^{(m)}}$ for a certain m-partition $\lambda^{(m)}$. Note that, given the knowledge of the values of q and v_j, $j = 1, \dots, m$, the basis $\{\mathcal{X}_{\lambda^{(m)}}\}$ is determined uniquely up to a global rescaling – if there were two bases then the operator, transforming one into another, would contradict to the irreducibility of the representation ρ. Now, the operation $A \mapsto \omega(A)^T$ is well-defined on matrices given by (4.16)-(4.17); for example, the matrix $\omega(\rho(\sigma_i))^T$ on the two-dimensional space with the basis $\{\mathcal{X}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}}\}$ is

$$\frac{1}{c^{(i+1)} - c^{(i)}} \begin{pmatrix} (q - q^{-1})c^{(i)} & qc^{(i+1)} - q^{-1}c^{(i)} \\ q^{-1}c^{(i+1)} - qc^{(i)} & -(q - q^{-1})c^{(i+1)} \end{pmatrix} , \tag{4.27}$$

with the same notation as in (4.18). The diagonal change of basis does not pose a problem either: in the product in (4.24), $\varsigma^{jk}_{X_{\lambda^{(m)}}}$ can be chosen as an arbitrary square root of the number $\dfrac{q^{-1}c^{(j)} - qc^{(k)}}{c^{(j)} - c^{(k)}}$, $(j,k) \in \mathfrak{E}_{X_{\lambda^{(m)}}}$. As a consequence, the matrices, in the basis $\{\tilde{\mathcal{X}}_{\lambda^{(m)}}\}$, of the generators of the affine Hecke algebra satisfy (4.25).

In this sense all irreducible finite-dimensional representations (from the described class) of the affine Hecke algebra are unitarizable.

Appendix 4.A Module structure on tensor products

This Appendix is rather technical and is not necessary for the understanding of the rest of the paper. It can be skipped at a first reading.

The algebra $\mathfrak{T}$, defined in Subsection 4.2, was used in Subsection 4.3 to construct modules over the cyclotomic Hecke algebra $H(m,1,n)$. An extension of this construction equips the tensor products of the underlying spaces of the $H(m,1,n)$-modules, corresponding to m-partitions of n, with a structure of an $H(m,1,n)$-module. In this Appendix we give precise definitions and investigate the appearing tensor product, denoted by $\hat{\otimes}$, of representations.

In Section 5 we prove that under the restrictions (2.10)–(2.12) on the parameters of $H(m,1,n)$, the irreducible representations of $H(m,1,n)$ are exhausted by the representations corresponding to m-partitions of n. Our method of studying the tensor product $\hat{\otimes}$ is inductive and is heavily based on the completeness result from Section 5. A priori, we do not know the nature of representations appearing in the decomposition of the tensor product $V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}}$ of two representations corresponding to m-partitions $\lambda^{(m)}$ and $\lambda'^{(m)}$. It is here that we need the heavy completeness result saying that every $H(m,1,n)$-module is isomorphic to a direct sum of $H(m,1,n)$-modules corresponding to m-partitions of n. Note that in its turn the completeness result is established in Section 5 indirectly, by counting dimensions.

The decomposition rules for the tensor product $\hat{\otimes}$ are given in the Proposition 10 in this Appendix. Qualitatively, the result is formulated very easily: the tensor product $V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}}$ is isomorphic to the direct sum of $\dim(V_{\lambda'^{(m)}})$ copies of the representation $V_{\lambda^{(m)}}$. For simplest choices of $\lambda^{(m)}$ and $\lambda'^{(m)}$, the decomposition of the tensor product $V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}}$ can be done by a direct calculation. However, in spite of the easiness of the formulation of the result of the Proposition 10, we could not find a way to perform an explicit calculation for two arbitrary m-partitions. Besides, rectangular partitions play a distinguished role in our way of proof of the Proposition 10 but not in the final formula for the decomposition rules. It would be interesting to find a more explicit way to establish the Proposition 10, without using the completeness assertion from Section 5.

By construction from subsection 4.3, the representation $V_{\lambda^{(m)}}$ corresponding to an m-partition $\lambda^{(m)}$ is equipped with the natural basis $\mathcal{X}_{\lambda^{(m)}}$ indexed by standard m-tableaux of the shape $\lambda^{(m)}$. The explicit form of an isomorphism $V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}} \cong \dim(V_{\lambda'^{(m)}})\, V_{\lambda^{(m)}}$ is quite evolved, showing again that the tensor product $\hat{\otimes}$ requires further understanding. We give several examples.

Certain statements below are valid in a more general situation, without the completeness assertion from Section 5. To accurately formulate these statements, we shall say that a representation of the cyclotomic Hecke algebra belongs to a class $\mathcal{S}$ if it is isomorphic to a direct sum of representations, corresponding to multi-partitions.

1. Definition of the tensor product $\hat{\otimes}$. Given an m-partition $\lambda^{(m)}$ of length n, recall that $U_{\lambda^{(m)}}$ is the vector space with the chosen basis $\{\mathcal{X}_{\lambda^{(m)}}\}$. We stress that $U_{\lambda^{(m)}}$ is understood only as a vector space, without any $H(m,1,n)$-module structure specified, whereas $V_{\lambda^{(m)}}$ is understood as the $H(m,1,n)$-module given by the formulas (4.16)-(4.17) with underlying vector space $U_{\lambda^{(m)}}$. In particular, a representation of $H(m,1,n)$ is of the class $\mathcal{S}$ if it is isomorphic to a direct sum of representations of the form $V_{\lambda^{(m)}}$.

Let $\lambda^{(m)}$ and $\lambda'^{(m)}$ be two m-partitions of length n. The instructions from the Proposition 8 are homogeneous in the generators $\mathcal{X}$. A basis of the tensor product of $U_{\lambda^{(m)}}$ and $U_{\lambda'^{(m)}}$ is naturally indexed by the products $\mathcal{X}_{\lambda^{(m)}}\mathcal{X}_{\lambda'^{(m)}}$, where $\mathcal{X}_{\lambda^{(m)}}$ is the generator labeled by the standard m-tableau $X_{\lambda^{(m)}}$ (of the shape $\lambda^{(m)}$) and $\mathcal{X}_{\lambda'^{(m)}}$ is the generator labeled by the standard m-tableau $X_{\lambda'^{(m)}}$ (of the shape $\lambda'^{(m)}$).

Moving now (following the instructions from the Proposition 8) in the expressions $\phi\mathcal{X}_{\lambda^{(m)}}\mathcal{X}_{\lambda'^{(m)}}|\rangle$, where $\phi \in H(m,1,n)$, the elements $\mathcal{X}$'s to the left and evaluating, with the help of (4.15), the elements of the cyclotomic algebra on the vacuum, we define the $H(m,1,n)$-module structure on the tensor product $U_{\lambda^{(m)}}\otimes U_{\lambda'^{(m)}}$ of the vector spaces underlying the representations $V_{\lambda^{(m)}}$ and $V_{\lambda'^{(m)}}$. We denote the resulting representation of $H(m,1,n)$ by $V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}}$.

In principle, the tensor product $\hat{\otimes}$ is defined for given m and n and should be rather denoted $\hat{\otimes}_{m,n}$. For brevity, we omit m in the notation for the product, the value of m is fixed in our treatment. As for n, below we introduce the operation of restriction from n to $(n-1)$ and explain that the tensor product $\hat{\otimes}$ is compatible with the restriction; due to the compatibility, we omit n in the notation for the product as well.

Notice that for any product $\mathcal{X}_{\lambda^{(m)}}\mathcal{X}_{\lambda'^{(m)}}$, the generator τ never passes through $\mathcal{X}_{\lambda^{(m)}}$ to the right (due to the particular form (4.9) of the instructions for the generator τ). Only the generators $\sigma_1, \dots, \sigma_{n-1}$ pass through $\mathcal{X}_{\lambda^{(m)}}$ and then act on $\mathcal{X}_{\lambda'^{(m)}}|\rangle$. The subalgebra of $H(m,1,n)$ generated by $\sigma_1, \dots, \sigma_{n-1}$ is isomorphic to the Hecke algebra $H(1,1,n)$ (this follows from the normal form from Appendix A, Corollary 39). Thus, it makes sense to consider tensor products of the form $V_{\lambda^{(m)}} \hat{\otimes} V$, where V is a representation of $H(1,1,n)$, as a representation of $H(m,1,n)$. Moreover, by construction, the representation $V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}}$ is naturally isomorphic to the representation $V_{\lambda^{(m)}} \hat{\otimes} W$, where W is the restriction of the representation $V_{\lambda'^{(m)}}$ to the subalgebra generated by $\sigma_1, \dots, \sigma_{n-1}$.

2. Product $\hat{\otimes}$: simplest examples. Let $\varpi^{(m)}$ be the m-partition $(\lambda_1, \dots, \lambda_m)$ of length n such that λ_1 is the one-row partition (n) and $\lambda_2, \dots, \lambda_m$ are empty partitions. There is only one standard m-tableau of shape $\varpi^{(m)}$ which we denote by $X_{\varpi^{(m)}}$. For this particular m-partition, the formulas (4.8)–(4.9) become:

$$(\sigma_i - q)\mathcal{X}_{\varpi^{(m)}} = 0 \quad \text{for all} \quad i = 1, \dots, n-1 \quad \text{and} \quad (\tau - v_1)\mathcal{X}_{\varpi^{(m)}} = 0 \ . \tag{4.28}$$

Thus the representation $V_{\varpi^{(m)}}$ is isomorphic to the one-dimensional representation of the algebra $H(m,1,n)$ spanned by the vacuum $|\rangle$. By construction, the following properties are verified (the isomorphisms are to be understood as isomorphisms of $H(m,1,n)$-modules): for any m-partition $\lambda^{(m)}$ of length n,

$$V_{\lambda^{(m)}} \hat{\otimes} V_{\varpi^{(m)}} \cong V_{\lambda^{(m)}} \ , \tag{4.29}$$

and

$$V_{\varpi^{(m)}} \hat{\otimes} V_{\lambda^{(m)}} \cong V_{\varpi^{(m)}} \oplus \dots \oplus V_{\varpi^{(m)}} \cong \dim(V_{\lambda^{(m)}}) \ V_{\varpi^{(m)}} \ . \tag{4.30}$$

Actually, in the formulas (4.29)–(4.30) one can replace $\varpi^{(m)}$ by any m-partition $\varpi'^{(m)}$ such that $V_{\varpi'^{(m)}}$ is one-dimensional; these are the m-partitions $(\lambda_1, ..., \lambda_m)$ with only one non-empty λ_j which equals (n) or (1^n). For the validity of (4.29) for $\varpi'^{(m)}$ see, for example, the Remark **(b)** after formulas (4.16)–(4.17); the validity of (4.30) for $\varpi'^{(m)}$ is immediate.

The formulas (4.29)–(4.30) are obtained in a straightforward manner. The Proposition 10 below describes the product $V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}}$ of representations corresponding to two arbitrary m-partitions of the same length. However the proof of the general formula (4.33) is not direct and relies on the completeness assertion which in turn requires the restrictions (2.10)–(2.12).

3. Restriction. An essential role in our study of the product $\hat{\otimes}$ is played by the operation of restriction which allows to use the induction arguments.

For any representation $\mathcal{W}$ of the algebra $H(m,1,n)$ we denote by $\mathrm{Res}^n_{n-1}(\mathcal{W})$ the restriction of $\mathcal{W}$ to the subalgebra of $H(m,1,n)$ generated by $\tau, \sigma_1, \dots, \sigma_{n-2}$; according to results from Appendix A (the assertion (ii) of the Proposition 38) this subalgebra is isomorphic to $H(m,1,n-1)$. This justifies the notation Res^n_{n-1}.

The class $\mathcal{S}$ of representations is stable with respect to the restriction Res^n_{n-1}. Indeed, the formulas (4.8)-(4.9) imply that for any m-partition $\lambda^{(m)}$ of length n,

$$\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}}) \cong \bigoplus_{\alpha^{(m)}:\ \alpha^{(m)} \in \mathcal{E}_-(\lambda^{(m)})} V_{\lambda^{(m)} \setminus \{\alpha^{(m)}\}} \ , \tag{4.31}$$

where, we recall, $\mathcal{E}_-(\lambda^{(m)})$ is the set of removable m-nodes of $\lambda^{(m)}$. The stability follows from (4.31).

Geometrically, it is clear that one can reconstruct the m-partition $\lambda^{(m)}$ from the set $\mathcal{E}_-(\lambda^{(m)})$ of its removable m-nodes. Therefore, by (4.31), for an irreducible $H(m,1,n)$-representation V, belonging to the class $\mathcal{S}$, $V \cong V_{\lambda^{(m)}}$, its restriction $\mathrm{Res}^n_{n-1}(V)$ characterizes the representation V of $H(m,1,n)$ uniquely up to isomorphism.

Moreover, a direct inspection shows that the operation $\hat{\otimes}$ on representations belonging to the class $\mathcal{S}$ is compatible with the operation of restriction in the following sense: for any two $H(m,1,n)$-representations $\mathcal{W}$ and $\mathcal{W}'$ belonging to $\mathcal{S}$, we have

$$\mathrm{Res}^n_{n-1}(\mathcal{W}\hat{\otimes}\mathcal{W}') \cong \big(\mathrm{Res}^n_{n-1}(\mathcal{W})\big)\hat{\otimes}\big(\mathrm{Res}^n_{n-1}(\mathcal{W}')\big) . \tag{4.32}$$

Here the symbol $\hat{\otimes}$ in the left hand side is the product for $H(m,1,n)$; in the right hand side it is the product for $H(m,1,n-1)$. The formula (4.32) justifies the usage of the symbol $\hat{\otimes}$ instead of more rigorous $\hat{\otimes}_n$.

Note that (4.31) and (4.32) are valid whenever the formula (4.8) makes sense for the participating representations (that is, the denominators in (4.8) do not vanish); we do not need here the completeness result from Section 5.

4. Decomposition rules. Under the constraints (2.10)–(2.12) on the parameters of $H(m,1,n)$, the product $\hat{\otimes}$ of two representations from the class $\mathcal{S}$ belongs again to the class $\mathcal{S}$ due to the completeness result from Section 5. The following Proposition gives the decomposition rules for the tensor product $\hat{\otimes}$ of the representations from the class $\mathcal{S}$.

Proposition 10. *Let $\lambda^{(m)}$ and $\lambda'^{(m)}$ be two arbitrary m-partitions of length n. Assume that the conditions (2.10)–(2.12) on the parameters of $H(m,1,n)$ hold. Then the representation $V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}}$ of $H(m,1,n)$ is isomorphic to the direct sum of $\dim(V_{\lambda'^{(m)}})$ copies of $V_{\lambda^{(m)}}$:*

$$V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}} \cong \dim(V_{\lambda'^{(m)}})\ V_{\lambda^{(m)}}. \tag{4.33}$$

Proof. We shall use here that two representations $V_{\pi_1^{(m)}}$ and $V_{\pi_2^{(m)}}$ of $H(m,1,n)$ corresponding to two different m-partitions $\pi_1^{(m)}$ and $\pi_2^{(m)}$ of n are non-isomorphic and that any representation of $H(m,1,n)$ belongs to the class $\mathcal{S}$ (this is proved, under the constraints (2.10)–(2.12), in Section 5).

We shall need the following Lemma.

Lemma 11. *(i) Let $\lambda^{(m)}$ be an m-partition of length n satisfying the following conditions:*

- *$\lambda^{(m)}$ is different from m-partitions of the form $(\varnothing,\dots,\varnothing,\lambda,\varnothing,\dots,\varnothing)$ where λ is a partition such that $|\lambda|\le 2$ or $\lambda=(2,1)$ or $\lambda=(2,1,1)$ or $\lambda=(3,1)$;*
- *$\lambda^{(m)}$ is different from m-partitions of the form $(\varnothing,\dots,\varnothing,\Box,\varnothing,\dots,\varnothing,\Box,\varnothing,\dots,\varnothing)$.*

(ii) Let $\mathfrak{L}=\{\lambda_1^{(m)},\dots,\lambda_l^{(m)}\}$ be any l-tuple of m-partitions of length n different from $\lambda^{(m)}$, $\lambda_j^{(m)}\neq\lambda^{(m)}$ for all $j=1,\dots,l$.

Then the two following sets of m-partitions

$$\lambda^{(m)-} := \big\{\lambda^{(m)}\backslash\{\alpha^{(m)}\}\big\}_{\alpha^{(m)}:\,\alpha^{(m)}\in\mathcal{E}_-(\lambda^{(m)})} \quad \textit{and} \quad \mathfrak{L}^- := \big\{\lambda_j^{(m)}\backslash\{\alpha_j^{(m)}\}\big\}_{j:\,j=1,\dots,l;\ \alpha_j^{(m)}:\,\alpha_j^{(m)}\in\mathcal{E}_-(\lambda_j^{(m)})}$$

do not coincide.

The Lemma 11 is a purely combinatorial statement. Its proof will be combinatorial as well. We shall prove that, if $\lambda^{(m)-}$ is contained in $\mathfrak{L}^-$ then there exists a sub-m-partition of length $n-1$ of one of $\lambda_j^{(m)} \in \mathfrak{L}$ which is not a sub-m-partition of $\lambda^{(m)}$. The representation-theoretic translation of the last sentence is the following.

Corollary 12. *Under the conditions (i) and (ii), if* $\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}})$ *is isomorphic to a sub-representation of* $\mathrm{Res}^n_{n-1}(V_{\lambda_1^{(m)}}\oplus\cdots\oplus V_{\lambda_l^{(m)}})$*, then there exists an m-partition* $\nu^{(m)}$ *of length* $n-1$ *such that* $V_{\nu^{(m)}}$ *is isomorphic to a sub-representation of* $\mathrm{Res}^n_{n-1}(V_{\lambda_1^{(m)}}\oplus\cdots\oplus V_{\lambda_l^{(m)}})$ *but not to a sub-representation of* $\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}})$*.*

Proof of the Lemma. Assume that the l-tuple $\mathfrak{L} = \{\lambda_1^{(m)},\dots,\lambda_l^{(m)}\}$, formed by m-partitions different from $\lambda^{(m)}$, is such that $\lambda^{(m)-}$ is contained in $\mathfrak{L}^-$. Then for each $\alpha^{(m)} \in \mathcal{E}_-(\lambda^{(m)})$ the m-partition $\lambda^{(m)}\backslash\{\alpha^{(m)}\}$ of $n-1$ is a sub-m-partition of some m-partition from $\mathfrak{L}$; there may be several m-partitions from $\mathfrak{L}$ with this property; choose one of them and denote it by $\mu^{(m)}$. By condition (ii), the m-partition $\mu^{(m)}$ is obtained from $\lambda^{(m)}\backslash\{\alpha^{(m)}\}$ by adding an m-node $\beta^{(m)} \in \mathcal{E}_+(\lambda^{(m)}\backslash\{\alpha^{(m)}\})$ different from $\alpha^{(m)}$.

For a removable m-node $\gamma^{(m)}$ of $\mu^{(m)}$, it is geometrically clear that the m-partition $\mu^{(m)}\backslash\{\gamma^{(m)}\}$ is not a sub-m-partition of $\lambda^{(m)}$ if and only if $\gamma^{(m)}$ is different from $\beta^{(m)}$; indeed, as for ordinary partitions, the inclusion graph (see Appendix B for the definition) of m-partitions is a lattice; in particular, it cannot contain a subgraph of the form

$$\begin{tikzcd} \lambda^{(m)}\backslash\{\alpha^{(m)}\} = \mu^{(m)}\backslash\{\beta^{(m)}\}\ \bullet \arrow[d, dash] \arrow[dr, dash] & \bullet\ \mu^{(m)}\backslash\{\gamma^{(m)}\} \arrow[dl, dash] \arrow[d, dash] \\ \lambda^{(m)}\ \bullet & \bullet\ \mu^{(m)} \end{tikzcd}$$

So, if $\mu^{(m)}$ has a removable m-node $\gamma^{(m)}$ different from $\beta^{(m)}$ then $\mu^{(m)}\backslash\{\gamma^{(m)}\}$ is a sub-m-partition of one of the $\lambda_j^{(m)} \in \mathfrak{L}$ and is not a sub-m-partition of $\lambda^{(m)}$. Therefore it suffices to show that there is an m-node $\alpha^{(m)} \in \mathcal{E}_-(\lambda^{(m)})$ such that, for any m-partition $\mu^{(m)}$ obtained by adding to $\lambda^{(m)}\backslash\{\alpha^{(m)}\}$ an m-node $\beta^{(m)} \in \mathcal{E}_+(\lambda^{(m)}\backslash\{\alpha^{(m)}\})$, different from $\alpha^{(m)}$, there is a removable m-node $\gamma^{(m)} \in \mathcal{E}_-(\mu^{(m)})$, different from $\beta^{(m)}$.

Such $\gamma^{(m)}$ does not exist if and only if the m-partition $\mu^{(m)}$ has only one removable m-node, that is,

$$\begin{array}{l}\text{only one partition in the } m\text{-tuple } \mu^{(m)} \text{ is non-empty}\\ \text{and this partition is of rectangular shape.}\end{array} \tag{4.34}$$

Thus, an m-partition $\lambda^{(m)}$ contradicting to the assertion of the Lemma 11 should verify the property: for every $\alpha^{(m)} \in \mathcal{E}_-(\lambda^{(m)})$ there exists $\beta^{(m)} \in \mathcal{E}_+(\lambda^{(m)}\backslash\{\alpha^{(m)}\})$, different from $\alpha^{(m)}$, such that the m-partition $\mu^{(m)}$, obtained by adding to $\lambda^{(m)}\backslash\{\alpha^{(m)}\}$ the m-node $\beta^{(m)}$, is described by (4.34). A direct inspection shows that such m-partitions are exactly those which are excluded by the part (i) of the formulation of the Lemma 11. $\square$

We now return to the proof of the Proposition 10.

We proceed by induction on n; the formula (4.33) is trivial for $n = 0$, that is, for $\lambda^{(m)} = (\varnothing, \dots, \varnothing)$. As the proof of the induction step shows we need to verify separately several cases to complete the proof of the Proposition 10.

Induction step. Let $\lambda^{(m)}$ and $\lambda'^{(m)}$ be two m-partitions of length n such that $\lambda^{(m)}$ satisfies the conditions from the part (i) of the Lemma 11. Due to the formulas (4.31)–(4.32),

$$\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}}) \cong \bigoplus_{\alpha^{(m)},\alpha'^{(m)} \ : \ \substack{\alpha^{(m)} \in \mathcal{E}_-(\lambda^{(m)}) \\ \alpha'^{(m)} \in \mathcal{E}_-(\lambda'^{(m)})}} V_{\lambda^{(m)}\backslash\{\alpha^{(m)}\}} \hat{\otimes} V_{\lambda'^{(m)}\backslash\{\alpha'^{(m)}\}} . \tag{4.35}$$

Our induction hypothesis is: the formula (4.33) is valid for the products in the right hand side of (4.35). Assuming the induction hypothesis, we transform the right hand side of (4.35),

$$\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}}) \cong \dim(V_{\lambda'^{(m)}})\ \mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}}). \tag{4.36}$$

Now we shall employ several times the completeness result from Section 5. First of all, the representation $V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}}$ belongs to $\mathcal{S}$ so we can write $V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}} = c\ V_{\lambda^{(m)}} \oplus W$ where

- W belongs to $\mathcal{S}$ and has no irreducible constituents isomorphic to $V_{\lambda^{(m)}}$,
- c is a non-negative integer; $c \le \dim(V_{\lambda'^{(m)}})$ by the dimension argument.

If $c < \dim(V_{\lambda'^{(m)}})$ we use again results of Section 5 to obtain a contradiction. Due to the semi-simplicity at level $n-1$ (note that the conditions (2.10)–(2.12) at level $n-1$ are implied by the conditions (2.10)–(2.12) at level n), the representation monoid of $H(m, 1, n-1)$ is cancellative; we can thus simplify (4.36) by $c\,\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}})$ on both sides. We obtain:

$$\mathrm{Res}^n_{n-1}(W) \cong \Big(\dim(V_{\lambda'^{(m)}}) - c\Big)\ \mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}})\ .$$

Since $\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}})$ is isomorphic to a sub-representation of $\mathrm{Res}^n_{n-1}(W)$, the Corollary 12 implies the existence of an m-partition $\nu^{(m)}$ of $n-1$ such that $\nu^{(m)}$ is not a sub-m-partition of $\lambda^{(m)}$ but $V_{\nu^{(m)}}$ is isomorphic to a sub-representation of $\mathrm{Res}^n_{n-1}(W)$. But now the representation $\mathrm{Res}^n_{n-1}(W)$ is isomorphic to a direct sum of several copies of $\mathrm{Res}^n_{n-1}(V_{\lambda^{(m)}})$ implying that such $\nu^{(m)}$ cannot exist, a contradiction. Thus $c = \dim(V_{\lambda'^{(m)}})$ and

$$V_{\lambda^{(m)}} \hat{\otimes} V_{\lambda'^{(m)}} \cong \dim(V_{\lambda'^{(m)}})\ V_{\lambda^{(m)}}.$$

The proof of the induction step is finished.

End of the proof of the Proposition 10. If the formula (4.33) is verified for an m-partition $\lambda^{(m)}$ and an arbitrary m-partition ${\lambda'}^{(m)}$ of the same length as $\lambda^{(m)}$, we shall simply say that (4.33) is verified for $\lambda^{(m)}$. Our way of proof of the induction step referred to the Lemma 11. So if the formula (4.33) is established for all m-partitions of length n then it is established for all m-partitions $\lambda^{(m)}$ of length $(n+1)$ unless $\lambda^{(m)}$ belongs to the set of m-partitions listed in part (i) of the Lemma 11. For the m-partitions listed in part (i) of the Lemma 11 an independent proof is needed. Besides, by (4.30), the formula (4.33) is already established for m-partitions $\lambda^{(m)}$ of 1, that is, for $\lambda^{(m)} = (\varnothing, \dots, \varnothing, \square, \varnothing, \dots, \varnothing)$ and for m-partitions of the form

$$(\varnothing, \dots, \varnothing, \lambda, \varnothing, \dots, \varnothing)\ , \ \text{ where } \lambda \text{ is } (2) \text{ or } (1,1)\ . \tag{4.37}$$

Below we shall separately verify that (4.33) holds for other m-partitions $\lambda^{(m)}$ listed in the part (i) of the Lemma 11, that is, m-partitions $\lambda^{(m)}$ of the form

$$(\varnothing, \dots, \varnothing, \lambda, \varnothing, \dots, \varnothing)\ , \ \text{ where } \lambda \text{ is } (2,1)\ ,\ (2,1,1) \text{ or } (3,1) \tag{4.38}$$

or of the form

$$(\varnothing, \dots, \varnothing, \square, \varnothing, \dots, \varnothing, \square, \varnothing, \dots, \varnothing)\ . \tag{4.39}$$

Proof of (4.33) for the m-partitions of the forms (4.38) and (4.39).

We recall here that, for any $\lambda^{(m)}$, ${\lambda'}^{(m)}$, the representation $V_{\lambda^{(m)}} \hat{\otimes} V_{{\lambda'}^{(m)}}$ is naturally isomorphic to a representation $V_{\lambda^{(m)}} \hat{\otimes} W$ where W is a representation of the Hecke algebra $H(1,1,n)$ (see the end of the paragraph **1** of this appendix). Due to the completeness result of Section 5, it is enough to consider the cases $W \cong V_\lambda$ for all partitions λ.

For the m-partitions $\lambda^{(m)}$ of the form (4.38), the generator τ acts by a constant in $V_{\lambda^{(m)}}$ and it is thus sufficient to work with the Hecke algebra $H(1,1,n)$.

Reintroduce for this paragraph the deformation parameter q in the notation for the Hecke algebra: $H_q(1,1,n)$. Due to the relations (2.1)–(2.2) and (2.7), we have an isomorphism $\theta\colon H_q(1,1,n) \to H_{-q^{-1}}(1,1,n)$ of algebras, defined on generators by $H_q(1,1,n) \ni \sigma_i \mapsto \sigma_i \in H_{-q^{-1}}(1,1,n)$. The composition with θ of a representation of $H_{-q^{-1}}(1,1,n)$, corresponding to a partition λ, sends the representations $V_{(3,1)}$, $V_{(2,2)}$ and $V_{(2,1,1)}$ of $H_{-q^{-1}}(1,1,n)$ to, respectively, the representations $V_{(2,1,1)}$, $V_{(2,2)}$ and $V_{(3,1)}$ of $H_q(1,1,n)$. Thus the formula (4.33) for $\lambda = (3,1)$ follows from the formula (4.33) for $\lambda = (2,1,1)$.

We remind that (4.33) has already been proved for any m-partitions $\lambda'^{(m)}$ such that $V_{\lambda'^{(m)}}$ is one-dimensional, see (4.30).

For the m-partitions $\lambda^{(m)}$ of the form (4.39), the proof of (4.33) is reduced to the situation where $V_{\lambda'^{(m)}}$ is replaced by $V_{\lambda'}$ where λ' is (2) or $(1,1)$ in which case the representation $V_{\lambda'}$ is one-dimensional and the formula (4.33) follows.

1. For the m-partition $\lambda^{(m)}$ of the form (4.38) with $\lambda = (2,1)$ we have reduced the proof to the situation $m=1$, and it remains to establish the result (4.33) only for $V_\lambda \hat{\otimes} V_\lambda$.

The underlying vector space of V_λ has a basis $\{\mathcal{X}_1, \mathcal{X}_2\}$ where $\mathcal{X}_1 := \mathcal{X}_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}}$ and $\mathcal{X}_2 := \mathcal{X}_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}}$.
In this basis, the generators σ_1 and σ_2 are realized as follows:

$$\sigma_1 \mapsto \mathrm{diag}(q, -q^{-1}) \ , \ \sigma_2 \mapsto \frac{1}{2_q} \begin{pmatrix} -q^{-2} & 3_q \\ 1 & q^2 \end{pmatrix} \ . \tag{4.40}$$

We order the basis $\mathcal{X}_i\mathcal{X}_j$ of the underlying vector space of $V_\lambda \hat{\otimes} V_\lambda$ lexicographically; that is, we choose the order $\{\mathcal{X}_1\mathcal{X}_1, \mathcal{X}_1\mathcal{X}_2, \mathcal{X}_2\mathcal{X}_1, \mathcal{X}_2\mathcal{X}_2\}$. In this basis, the matrices of the generators σ_1 and σ_2 are:

$$\sigma_1 \mapsto \mathrm{diag}(q, q, -q^{-1}, -q^{-1}) \ , \ \sigma_2 \mapsto \frac{1}{2_q} \begin{pmatrix} -q^{-2} & 0 & 0 & 3_q \\ 0 & -q^{-2} & 1 & q^2 + q^{-2} \\ -q^2 - q^{-2} & 3_q & q^2 & 0 \\ 1 & 0 & 0 & q^2 \end{pmatrix} \ .$$

The two subspaces with the bases

$$\begin{gathered} \{\mathcal{X}_1\mathcal{X}_2, 3_q\mathcal{X}_2\mathcal{X}_1\} \ , \\ \{\mathcal{X}_1\mathcal{X}_1 + \mathcal{X}_1\mathcal{X}_2, \mathcal{X}_2\mathcal{X}_1 + \mathcal{X}_2\mathcal{X}_2\} \end{gathered} \tag{4.41}$$

carry the representation (4.40) which implies (4.33) in this case. Since $V_\lambda \hat{\otimes} V_\lambda$ decomposes into a direct sum of two isomorphic representations, the choice (4.41) of subspaces is not unique. We just make (here and for the other cases below) a simple choice.

2. For the m-partition $\lambda^{(m)}$ of the form (4.38) with $\lambda = (2, 1, 1)$ we have reduced the proof to the situation $m = 1$, and it remains to establish the result (4.33) only for $V_\lambda \hat{\otimes} V_{\lambda'}$ with $\lambda' = (2, 2), (2, 1, 1)$ and $(3, 1)$.

The underlying vector space of V_λ has a basis $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$ where $\mathcal{X}_1 := \mathcal{X}_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}}$, $\mathcal{X}_2 := \mathcal{X}_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}}$
and $\mathcal{X}_3 := \mathcal{X}_{\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}}$. In this basis, the generators σ_1, σ_2 and σ_3 are realized as follows:

$$\begin{gathered} \sigma_1 \mapsto \mathrm{diag}(q, -q^{-1}, -q^{-1}) \ , \\ \sigma_2 \mapsto \tfrac{1}{2_q} \begin{pmatrix} -q^{-2} & 3_q & 0 \\ 1 & q^2 & 0 \\ 0 & 0 & -q^{-1}2_q \end{pmatrix} \ , \ \sigma_3 \mapsto \tfrac{1}{3_q} \begin{pmatrix} -q^{-1}3_q & 0 & 0 \\ 0 & -q^{-3} & 4_q \\ 0 & 2_q & q^3 \end{pmatrix} \ . \end{gathered} \tag{4.42}$$

2a. $\lambda' = (2, 2)$.

The underlying vector space of $V_{\lambda'}$ has a basis $\{\mathcal{Y}_1, \mathcal{Y}_2\}$, where $\mathcal{Y}_1 := \mathcal{X}_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}}$ and $\mathcal{Y}_2 := \mathcal{X}_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}}$.
We order the basis $\mathcal{X}_i\mathcal{Y}_j$ of the underlying vector space of $V_\lambda \hat{\otimes} V_{\lambda'}$ lexicographically. In this basis the

generators σ_1, σ_2 and σ_3 are realized as follows:

$$\sigma_1 \mapsto \mathrm{diag}(q, q, -q^{-1}, -q^{-1}, -q^{-1}, -q^{-1}) \ ,$$

$$\sigma_2 \mapsto \frac{1}{2_q} \begin{pmatrix} -q^{-2} & 0 & 0 & 3_q & 0 & 0 \\ 0 & -q^{-2} & 1 & q^2 + q^{-2} & 0 & 0 \\ -q^2 - q^{-2} & 3_q & q^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^{-1}2_q & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1}2_q \end{pmatrix} ,$$

$$\sigma_3 \mapsto \frac{1}{3_q} \begin{pmatrix} -q^{-1}3_q & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^{-1}3_q & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-3} & 0 & 4_q & 0 \\ 0 & 0 & 0 & -q^{-3} & 0 & -2_q \\ 0 & 0 & 2_q & 0 & q^3 & 0 \\ 0 & 0 & 0 & -4_q & 0 & q^3 \end{pmatrix} ,$$

The two subspaces with the bases

$$\{\mathcal{X}_1\mathcal{Y}_2, 3_q\mathcal{X}_2\mathcal{Y}_1, 3_q\mathcal{X}_3\mathcal{Y}_1\} \ ,$$

$$\{\mathcal{X}_1\mathcal{Y}_1 + \mathcal{X}_1\mathcal{Y}_2, \mathcal{X}_2\mathcal{Y}_1 + \mathcal{X}_2\mathcal{Y}_2, \mathcal{X}_3\mathcal{Y}_1 - (q^2 + q^{-2})\mathcal{X}_3\mathcal{Y}_2\}$$

carry the representation (4.42), which implies (4.33) in this case.

2b. $\lambda' = \lambda = (2, 1, 1)$.

We order the basis $\mathcal{X}_i\mathcal{X}_j$ of the underlying vector space of $V_\lambda \hat{\otimes} V_\lambda$ lexicographically. In this basis the generators σ_1, σ_2 and σ_3 are realized as follows:

$$\sigma_1 \mapsto \mathrm{diag}(q, q, q, -q^{-1}, -q^{-1}, -q^{-1}, -q^{-1}, -q^{-1}, -q^{-1}) \ ,$$

$$\sigma_2 \mapsto \frac{1}{2_q} \begin{pmatrix} -q^{-2} & 0 & 0 & 0 & 3_q & 0 & 0 & 0 & 0 \\ 0 & -q^{-2} & 0 & 1 & q^2 + q^{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-2} & 0 & 0 & -1 & 0 & 0 & 0 \\ -q^2 - q^{-2} & 3_q & 0 & q^2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3_q & 0 & 0 & q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}2_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}2_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}2_q \end{pmatrix} ,$$

$$\sigma_3 \mapsto \frac{1}{3_q}\begin{pmatrix} -q^{-1}3_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^{-1}3_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1}3_q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^{-3} & 0 & 0 & -2_q & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^{-3} & 0 & 0 & 0 & 4_q \\ 0 & 0 & 0 & 0 & 0 & -q^{-3} & 0 & 2_q & q^3+q^{-3} \\ 0 & 0 & 0 & -4_q & 0 & 0 & q^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q^3-q^{-3} & 4_q & 0 & q^3 & 0 \\ 0 & 0 & 0 & 0 & 2_q & 0 & 0 & 0 & q^3 \end{pmatrix} ,$$

The three subspaces with the bases

$$\{\mathcal{X}_1\mathcal{Y}_2, 3_q\mathcal{X}_2\mathcal{Y}_1, -3_q(q^2+q^{-2})\mathcal{X}_3\mathcal{Y}_1\} ,$$

$$\{\mathcal{X}_1\mathcal{Y}_1+\mathcal{X}_1\mathcal{Y}_2, \mathcal{X}_2\mathcal{Y}_1+\mathcal{X}_2\mathcal{Y}_2, -(q^2+q^{-2})\mathcal{X}_3\mathcal{Y}_1 - \frac{q^3+q^{-3}}{2_q}\mathcal{X}_3\mathcal{Y}_2+\mathcal{X}_3\mathcal{Y}_3\} ,$$

$$\{\mathcal{X}_1\mathcal{Y}_3, -3_q\mathcal{X}_2\mathcal{Y}_3, -3_q(q^2+q^{-2})\mathcal{X}_3\mathcal{Y}_2\}$$

carry the representation (4.42), which implies (4.33) in this case.

2c. $\lambda' = (3,1)$.

The underlying vector space of $V_{\lambda'}$ has a basis $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$, where $\mathcal{Y}_1 := \mathcal{X}_{\tiny\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}}$, $\mathcal{Y}_2 := \mathcal{X}_{\tiny\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}}$ and $\mathcal{Y}_3 := \mathcal{X}_{\tiny\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}}$. We order the basis $\mathcal{X}_i\mathcal{Y}_j$ of the underlying vector space of $V_\lambda\hat{\otimes}V_{\lambda'}$ lexicographically. In this basis the generators σ_1, σ_2 and σ_3 are realized as follows:

$$\sigma_1 \mapsto \mathrm{diag}(q,q,q,-q^{-1},-q^{-1},-q^{-1},-q^{-1},-q^{-1},-q^{-1}) ,$$

$$\sigma_2 \mapsto \frac{1}{2_q}\begin{pmatrix} -q^{-2} & 0 & 0 & q^2+q^{-2} & 1 & 0 & 0 & 0 & 0 \\ 0 & -q^{-2} & 0 & 3_q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-2} & 0 & 0 & 3_q & 0 & 0 & 0 \\ 0 & 1 & 0 & q^2 & 0 & 0 & 0 & 0 & 0 \\ 3_q & -q^2-q^{-2} & 0 & 0 & q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}2_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}2_q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1}2_q \end{pmatrix} ,$$

$$\sigma_3 \mapsto \frac{1}{3_q}\begin{pmatrix} -q^{-1}3_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -q^{-1}3_q & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -q^{-1}3_q & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -q^{-3} & 0 & 0 & 4_q & 0 & 0\\ 0 & 0 & 0 & 0 & -q^{-3} & 0 & 0 & q^3+q^{-3} & 2_q\\ 0 & 0 & 0 & 0 & 0 & -q^{-3} & 0 & 4_q & 0\\ 0 & 0 & 0 & 2_q & 0 & 0 & q^3 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 2_q & 0 & q^3 & 0\\ 0 & 0 & 0 & 0 & 4_q & -q^3-q^{-3} & 0 & 0 & q^3 \end{pmatrix} ,$$

The three subspaces with the bases

$$\{\mathcal{X}_1\mathcal{Y}_1, 3_q\mathcal{X}_2\mathcal{Y}_2, 3_q(q^2+q^{-2})\mathcal{X}_3\mathcal{Y}_3\}\ ,$$

$$\{\mathcal{X}_1\mathcal{Y}_1+\mathcal{X}_1\mathcal{Y}_2, \mathcal{X}_2\mathcal{Y}_1+\mathcal{X}_2\mathcal{Y}_2, \mathcal{X}_3\mathcal{Y}_1+(q^2+q^{-2})\mathcal{X}_3\mathcal{Y}_3\}\ ,$$

$$\{\mathcal{X}_1\mathcal{Y}_3, \mathcal{X}_2\mathcal{Y}_3, \mathcal{X}_3\mathcal{Y}_2-\frac{q^3+q^{-3}}{2_q}\mathcal{X}_3\mathcal{Y}_3\}$$

carry the representation (4.42), which implies (4.33) in this case.

The proof of the Proposition 10 is completed. □

Remarks.

(a) By the Proposition 10, the tensor product $\hat{\otimes}$ is obviously associative. We remark that if we had an independent proof of the associativity of $\hat{\otimes}$ it would immediately imply the Proposition 10:

$$V_{\lambda^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}} \cong (V_{\lambda^{(m)}}\hat{\otimes}V_{\varpi^{(m)}})\hat{\otimes}V_{\lambda'^{(m)}} \cong V_{\lambda^{(m)}}\hat{\otimes}(V_{\varpi^{(m)}}\hat{\otimes}V_{\lambda'^{(m)}}) \cong \dim(V_{\lambda'^{(m)}})\ V_{\lambda^{(m)}}\ ;$$

we used (4.29) for the first isomorphism and (4.29)-(4.30) for the last one.

(b) Let $\lambda^{(m)}$ be an m-partition of length n and $\rho\colon H(m,1,n)\to \mathrm{End}(\mathfrak{V})$ a representation of $H(m,1,n)$. One can construct a representation of $H(m,1,n)$ on the space $U_{\lambda^{(m)}}\otimes\mathfrak{V}$ by moving the elements of $H(m,1,n)$ through the basis elements $\mathcal{X}_{\lambda^{(m)}}$ of $U_{\lambda^{(m)}}$ with the help of instructions from the Proposition 8 and then applying the representation ρ. Denote this representation by $V_{\lambda^{(m)}}\boxtimes\mathfrak{V}$. If ρ is a representation from the class S, this construction is equivalent to the tensor product $\hat{\otimes}$,

$$V_{\lambda^{(m)}}\boxtimes\mathfrak{V}\cong V_{\lambda^{(m)}}\hat{\otimes}\mathfrak{V}\ . \tag{4.43}$$

Similarly to the construction of a $H(m,1,n)$-module structure on the space $U_{\lambda^{(m)}}\otimes U_{\lambda'^{(m)}}$, one can construct a $H(m,1,n)$-module structure on the tensor product of spaces corresponding to l arbitrary m-partitions, with $l\in\mathbb{Z}_{\geq 0}$. One has to replace in the construction the quadratic combinations $\mathcal{X}_{\lambda^{(m)}}\mathcal{X}_{\lambda'^{(m)}}$ by combinations $\mathcal{X}_{\lambda_l^{(m)}}\mathcal{X}_{\lambda_{l-1}^{(m)}}\dots\mathcal{X}_{\lambda_1^{(m)}}$ of degree l, move the elements of $H(m,1,n)$ through these combinations using the homogeneous (in $\mathcal{X}$) relations (4.8)–(4.9) and then evaluating them on the vacuum $|\rangle$. Denote this representation by $\mathfrak{W}_{\lambda_l^{(m)},\lambda_{l-1}^{(m)},\dots,\lambda_1^{(m)}}$. We claim that $\mathfrak{W}_{\lambda_l^{(m)},\lambda_{l-1}^{(m)},\dots,\lambda_1^{(m)}}$ is isomorphic to the direct sum of $\dim(V_{\lambda_1^{(m)}}) \times \cdots \times \dim(V_{\lambda_{l-1}^{(m)}})$ copies of $V_{\lambda_l^{(m)}}$. Indeed, by (4.43), $\mathfrak{W}_{\lambda_l^{(m)},\lambda_{l-1}^{(m)},\dots,\lambda_1^{(m)}}$ is equivalent to the representation $V_{\lambda_l^{(m)}} \boxtimes \mathfrak{W}_{\lambda_{l-1}^{(m)},\dots,\lambda_1^{(m)}}$. By induction (the induction base is the formula (4.33)) the representation $\mathfrak{W}_{\lambda_{l-1}^{(m)},\dots,\lambda_1^{(m)}}$ is isomorphic to the direct sum of $\dim(V_{\lambda_1^{(m)}}) \times \cdots \times \dim(V_{\lambda_{l-2}^{(m)}})$ copies of $V_{\lambda_{l-1}^{(m)}}$. By (4.33) and (4.43), the claim follows.

(c) The partition $(2,1,1)$ appears in the list from the part (i) of the Lemma 11 because

$$\mathrm{Res}_3^4(V_{(2,1,1)}) \cong \mathrm{Res}_3^4(V_{(2,2)}) \oplus \mathrm{Res}_3^4(V_{(1,1,1,1)}) \ .$$

For the representation $V_{(2,1,1)}$ the matrix of the operator $\sigma_1\sigma_3$ is

$$\frac{1}{3_q}\begin{pmatrix} -3_q & 0 & 0 \\ 0 & q^{-4} & -q^{-1}4_q \\ 0 & -q^{-1}2_q & -q^2 \end{pmatrix} \ .$$

Thus, $\mathrm{tr}_{V_{(2,1,1)}}(\sigma_1\sigma_3) = -2 + q^{-2}$. In the representation $V_{(1,1,1,1)}$ we have $\sigma_1, \sigma_3 \mapsto (-q^{-1})$ while in the representation $V_{(2,2)}$ we have $\sigma_1, \sigma_3 \mapsto \mathrm{diag}(q, -q^{-1})$, so $\mathrm{tr}_{V_{(1,1,1,1)}}(\sigma_1\sigma_3) + \mathrm{tr}_{V_{(2,2)}}(\sigma_1\sigma_3) = q^2 + 2q^{-2}$. This differs from $\mathrm{tr}_{V_{(2,1,1)}}(\sigma_1\sigma_3) = -2 + q^{-2}$ if and only if $(q+q^{-1})^2 \neq 0$ that is, $q + q^{-1} \neq 0$. Therefore to establish the formula (4.33) for the representation $V_{(2,1,1)}$ it is enough to calculate the trace of $\sigma_1\sigma_3$ in the representations $V_{(2,1,1)}\hat{\otimes} V_\lambda$ with $\lambda = (2,2), (2,1,1)$ and $(3,1)$. Note that this argument works, in particular, in the classical limit $q \to 1$.

5 Completeness

1. In the preceding Section we constructed, for every m-partition $\lambda^{(m)}$, a representation of $H(m,1,n)$. The spectrum of the Jucys–Murphy elements $J_1, \dots, J_n$ in this representation is the set of strings corresponding to the standard m-tableaux of shape $\lambda^{(m)}$, see the Lemma 9. This construction provides an inclusion of the set of standard Young m-tableaux of length n into $\mathrm{Spec}(J_1, \dots, J_n)$. On the other hand, the Proposition 3 and the Proposition 5 provide an inclusion of $\mathrm{Spec}(J_1, \dots, J_n)$ into the set of standard Young m-tableaux of length n. These operations, by construction, are inverse to each other. We sum up the results.

We underline that the restrictions (2.10)–(2.12) are essential for the statements below.

Proposition 13. *The set* $\mathrm{Spec}(J_1, \dots, J_n)$*, the set* $\mathrm{Cont}_m(n)$ *and the set of standard* m*-tableaux are in bijection.*

Corollary 14. *The spectrum of the Jucys–Murphy elements is simple in the representations* $V_{\lambda^{(m)}}$ *(labeled by the* m*-partitions).*

It means that for two different standard m-tableaux (not necessarily of the same shape) the elements of $\mathrm{Spec}(J_1, \dots, J_n)$ associated to them by the Proposition 13 are different (two strings $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$ are different if there is some i such that $a_i \neq a'_i$).

2. It remains to verify that we obtain within this approach all irreducible representations of the algebra $H(m,1,n)$.

According to Appendix B the sum of the squares of the dimensions of the constructed representations equals the dimension of $H(m,1,n)$. It is therefore sufficient to prove that these representations are irreducible and pairwise non-isomorphic. It is done in the Theorem 15.

As a by-product we obtain that the algebra $H(m,1,n)$ is semi-simple under the restrictions (2.10)–(2.12).

Theorem 15. *The representations $V_{\lambda^{(m)}}$ (labeled by the m-partitions) of the algebra $H(m,1,n)$ constructed in Section 4 are irreducible and pairwise non-isomorphic.*

Proof. The proof can be found in [1] (as well as the semi-simplicity result). We briefly repeat the argument for completeness.

In the proof we use induction on n. It is justified since the restrictions (2.10)–(2.12) for $H(m,1,n)$ imply the restrictions (2.10)–(2.12), in which n is replaced by n', for $H(m,1,n')$ with arbitrary n', $0<n'<n$.

The Corollary 14 directly implies that the representations $V_{\lambda^{(m)}}$ and $V_{\lambda'^{(m)}}$ are non-isomorphic if $\lambda^{(m)}\neq\lambda'^{(m)}$.

Suppose by induction that the representations $V_{\mu^{(m)}}$, for all m-partitions $\mu^{(m)}$ of $n-1$, are irreducible representations of $H(m,1,n-1)$. The base of induction is $n=1$; there is nothing to prove here.

Fix an m-partition $\lambda^{(m)}$ with $|\lambda^{(m)}|=n$. Let $\{\mu_i^{(m)}\}$, $i=1,\dots,l$, be the set of all m-sub-partitions of $\lambda^{(m)}$ with $|\mu_i^{(m)}|=n-1$.

For each i, the representation $V_{\mu_i^{(m)}}$ of $H(m,1,n-1)$ is a sub-representation of the restriction of the representation $V_{\lambda^{(m)}}$ to $H(m,1,n-1)$. The dimension of $V_{\lambda^{(m)}}$ (the number of standard m-tableaux of the shape $\lambda^{(m)}$) is the sum (over i) of dimensions of $V_{\mu_i^{(m)}}$. Therefore, the representation $V_{\lambda^{(m)}}$ of $H(m,1,n)$ decomposes with respect to $H(m,1,n-1)$ into a direct sum of representations $V_{\mu_i^{(m)}}$.

The m-sub-partitions $\mu_i^{(m)}$ are different and correspond thus to non-isomorphic irreducible representations of $H(m,1,n-1)$. It follows that the positions of $V_{\mu_i^{(m)}}$ as subspaces in $V_{\lambda^{(m)}}$ are well-defined. Therefore, if $V_{\lambda^{(m)}}$ has a non-trivial invariant subspace U then U must contain at least one of the $V_{\mu_i^{(m)}}$, say $V_{\mu_1^{(m)}}$.

It is sufficient to show that starting from elements of $V_{\mu_1^{(m)}}$ one can obtain an element of $V_{\mu_j^{(m)}}$ for any $j\neq 1$ by the action of operators from $H(m,1,n)$. A basis vector of $V_{\lambda^{(m)}}$ labeled by a standard m-tableau $X_{\lambda^{(m)}}$ of shape $\lambda^{(m)}$ belongs to the subspace $V_{\mu_j^{(m)}}$ where $\mu_j^{(m)}$ is the m-sub-partition of length $(n-1)$ formed by the m-nodes with $1,\dots,n-1$ of $X_{\lambda^{(m)}}$. For any $j\neq 1$ the m-sub-partition $\mu_j^{(m)}$ is obtained from $\mu_1^{(m)}$ by removing one m-node and adding some other m-node, different from the removed one; it is easy to see that the two m-nodes involved are non-adjacent and, even more, are not on neighboring diagonals. Take the standard m-tableau of shape $\lambda^{(m)}$ for which the numbers $1,\dots,n-1$ are placed in the m-sub-partition $\mu_1^{(m)}$ of $\lambda^{(m)}$ and moreover the number $n-1$ is in the only m-node of $\mu_1^{(m)}$ which is not in the m-sub-partition $\mu_j^{(m)}$. The vector $\vec{v}$ of $V_{\lambda^{(m)}}$ labeled by this m-tableau belongs to the subspace $V_{\mu_1^{(m)}}$ and is sent by σ_{n-1} to a combination of the vector $\vec{v}$ and a vector belonging to $V_{\mu_j^{(m)}}$. The formula (4.16) shows that this vector of $V_{\mu_j^{(m)}}$ is non-zero. □

Let $\mathcal{B}$ be an associative subalgebra of an associative algebra $\mathcal{A}$. An indecomposable (irreducible if the algebra $\mathcal{A}$ is semi-simple) representation of the algebra $\mathcal{A}$ "branches" with respect to the algebra $\mathcal{B}$, that is, decomposes into a direct sum of indecomposable (irreducible if the algebra $\mathcal{B}$ is semi-simple) representations of $\mathcal{B}$. The information about branchings of all representations of the algebra $\mathcal{A}$ with respect to the subalgebra $\mathcal{B}$ is called *branching rules* for the pair $(\mathcal{A},\mathcal{B})$.

As a corollary of the whole construction we obtain under the restrictions (2.10)–(2.12) the branching rules for the pair $\big(H(m,1,n),H(m,1,n-1)\big)$; the representation of the algebra $H(m,1,n)$ labeled by an m-partition $\lambda^{(m)}$ of n decomposes into the direct sum of the representations of the algebra $H(m,1,n)$ labeled by the m-sub-partitions of $\lambda^{(m)}$ of length $n-1$. In particular we obtain the following Corollary.

Corollary 16. *Under the restrictions (2.10)–(2.12) the branching rules for the chain, with respect to n, of the algebras $H(m,1,n)$ are multiplicity-free.*

It means that under the restrictions (2.10)–(2.12) in the decomposition of an irreducible representation of the algebra $H(m,1,n)$ each irreducible representation of the algebra $H(m,1,n-1)$ appears with the multiplicity equal either to 0 or to 1.

By the general arguments it follows that under the restrictions (2.10)–(2.12) the centralizer of the sub-algebra $H(m,1,n-1)$ in the algebra $H(m,1,n)$ is commutative for each $n=1,2,3,\dots$

It also follows from the constructed representation theory that under the restrictions (2.10)–(2.12)

- the centralizer of the subalgebra $H(m,1,n-1)$ in the algebra $H(m,1,n)$ is generated by the center of $H(m,1,n-1)$ and the Jucys–Murphy element J_n;
- the subalgebra generated by the Jucys–Murphy elements $J_1,\dots,J_n$ of the algebra $H(m,1,n)$ is maximal commutative.

Remarks.

(a) For every standard m-tableau $X_{\lambda^{(m)}}$ define the element $\mathfrak{P}_{X_{\lambda^{(m)}}}$ of the cyclotomic Hecke algebra $H(m,1,n)$ by the following recursion. The initial condition is $\mathfrak{P}_\varnothing=1$. Let $\alpha^{(m)}$ be the m-node occupied by the number n in $X_{\lambda^{(m)}}$; define $\mu^{(m)}:=\lambda^{(m)}\backslash\{\alpha^{(m)}\}$. Denote by $X_{\mu^{(m)}}$ the standard m-tableau with the numbers $1,\dots,n-1$ at the same m-nodes as in $X_{\lambda^{(m)}}$. Then the recursion is given by

$$\mathfrak{P}_{X_{\lambda^{(m)}}}:=\mathfrak{P}_{X_{\mu^{(m)}}}\prod_{\beta^{(m)}:\,\beta^{(m)}\in\mathcal{E}_+(\mu^{(m)}),\,\beta^{(m)}\neq\alpha^{(m)}}\frac{J_n-c(\beta^{(m)})}{c(\alpha^{(m)})-c(\beta^{(m)})}\tag{5.1}$$

where $c(\beta^{(m)})$ is the content of the m-node $\beta^{(m)}$. Due to the completeness results of this Section, the elements $\mathfrak{P}_{X_{\lambda^{(m)}}}$ form a complete set of pairwise orthogonal primitive idempotents of the algebra $H(m,1,n)$.

We shall prove that, moreover, we have a well-defined homomorphism $\varrho\colon\ \mathfrak{T}\to H(m,1,n)$ which is identical on the generators $\tau,\sigma_1,\dots,\sigma_{n-1}$ and sends $\mathcal{X}_{\lambda^{(m)}}$ to $\mathfrak{P}_{X_{\lambda^{(m)}}}$ for all standard m-tableaux $X_{\lambda^{(m)}}$. Using the completeness, the only non-trivial verification one has to do is to check that, for any standard m-tableau $X_{\lambda^{(m)}}$ such that $X^{s_i}_{\lambda^{(m)}}$ is standard, the defining relation (4.8) of the algebra $\mathfrak{T}$ is satisfied by the images of σ_i, $\mathcal{X}_{\lambda^{(m)}}$ and $\mathcal{X}^{s_i}_{\lambda^{(m)}}$ through the homomorphism ϱ. The verification reduces to the following equality for matrices (see (4.18)):

$$\begin{pmatrix} 0 & A \\ B & -C \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C & A \\ B & 0 \end{pmatrix}\ , \tag{5.2}$$

where $A=\dfrac{qc^{(i)}-q^{-1}c^{(i+1)}}{c^{(i)}-c^{(i+1)}}$, $B=\dfrac{qc^{(i+1)}-q^{-1}c^{(i)}}{c^{(i+1)}-c^{(i)}}$, $C=\dfrac{(q-q^{-1})(c^{(i+1)}+c^{(i)})}{c^{(i+1)}-c^{(i)}}$ and $c^{(i)}:=c(X_{\lambda^{(m)}}|i)$ for all $i=1,\dots,n$. The elements $\varrho(\mathcal{X}_{\lambda^{(m)}})$ are the diagonal matrix units; the elements

$$\varrho\Big(\sigma_i+\frac{(q-q^{-1})c(X_{\lambda^{(m)}}|i+1)}{c(X_{\lambda^{(m)}}|i)-c(X_{\lambda^{(m)}}|i+1)}\Big)\ \varrho(\mathcal{X}_{\lambda^{(m)}})=\varrho(\mathcal{X}^{s_i}_{\lambda^{(m)}})\varrho\Big(\sigma_i+\frac{(q-q^{-1})c(X_{\lambda^{(m)}}|i)}{c(X_{\lambda^{(m)}}|i+1)-c(X_{\lambda^{(m)}}|i)}\Big) \tag{5.3}$$

form a part of off-diagonal (non-normalized) matrix units - the calculation (5.2) shows that these elements are non-zero. For the usual Hecke algebra $H(1,1,n)$ the equality (5.3) was established in [24]. The complete set of off-diagonal matrix units was constructed in [25] for the usual Hecke algebra. The construction for the cyclotomic Hecke algebra is similar and we leave details to the reader.

(b) For a subset $\{v_{i_1},\dots,v_{i_l}\}$ with $l<m$ let $\mathfrak{z}:=(\tau-v_{i_1})...(\tau-v_{i_l})$. Taking a quotient of $H(m,1,n)$ by the ideal generated by $\mathfrak{z}$ we get a homomorphism $\mathfrak{p}\colon H(m,1,n)\to H(l,1,n)$ where $H(l,1,n)$ is the cyclotomic Hecke algebra with the parameters q, $v_{i_1},\dots,v_{i_l}$ (note that the restrictions (2.10)–(2.12) hold for this choice of the parameters if they hold for q, $v_1,\dots,v_n$). The representations of $H(m,1,n)$ for which the diagonal entries of the (diagonal) matrix (4.17) belong to $\{v_{i_1},\dots,v_{i_l}\}$ (these representations are labeled by m-partitions with empty partitions on the places corresponding to v_j which are omitted in $\{v_{i_1},\dots,v_{i_l}\}$) pass through the image $\mathfrak{p}(H(m,1,n))$ in $H(m,1,n)$. The sum of squares of dimensions of these representations equals to the dimension of $H(l,1,n)$. It follows that $\mathfrak{p}$ is surjective.

6. The classical limit

Here we consider the classical limit of the cyclotomic Hecke algebra $H(m,1,n)$, that is the group ring of the complex reflection group $G(m,1,n)$. The representation theory of the groups $G(m,1,n)$ is well known, see, [35] or, *e.g.*, [22]. Also, the representation theory of $G(m,1,n)$ can be directly deduced from the representation theory of $H(m,1,n)$ by taking the limit

$$\left\{\begin{array}{ll} v_i\to\xi_i & \text{for } i=1,\dots,m, \text{ where the } \xi_i \text{ are distinct } m^{\text{th}} \text{ roots of unity }, \\ q\to\pm1 & \end{array}\right. \tag{6.1}$$

in formulas for matrix elements. However it is interesting to take the "classical limit" of the whole above developed approach establishing thereby an approach to the representation theory of the group $G(m,1,n)$ not referring to the representation theory of $H(m,1,n)$. The construction of an algebra structure on the tensor product of the algebra $\mathbb{C}G(m,1,n)$ with a free associative algebra generated by the standard m-tableaux corresponding to m-partitions of n is of independent interest.

The representation theory of a more general class of groups, namely, of the wreath products of finite groups by the symmetric groups, was built, in the spirit of [28], in [29]. The construction in [29] is worked out within the group theory. In this section we shall see how this approach is restored – on the example of the groups $G(m,1,n)$, the wreath products of the cyclic groups by the symmetric groups – in the classical limit of the construction developed in the preceding sections for $H(m,1,n)$. We shall see that there are certain subtleties in passing to the classical "group" situation (one should be careful about the order of taking limits *etc*). As it often happens the classical situation is more complicated than the quantum one.

We more or less repeat the same steps as in the non-degenerate situation. We present the classical Jucys–Murphy elements of the group ring of $G(m,1,n)$, which we obtain as classical limits of certain expressions involving the Jucys-Murphy elements of $H(m,1,n)$; the Jucys–Murphy of the group ring of $G(m,1,n)$ are images of the "universal" Jucys–Murphy elements living in a version of a degenerate cyclotomic affine Hecke algebra (in contrast with the non-degenerate situation where we need the usual affine Hecke algebra), which we denote $\mathfrak{A}_{m,n}$. The algebra $\mathfrak{A}_{m,n}$ turns out to coincide with a particular case of the wreath Hecke algebra defined in [39] (see also [31]). To study the spectrum of the Jucys–Murphy of the group ring of $G(m,1,n)$ we use representations of the simplest non-trivial degenerate cyclotomic affine Hecke algebra, the algebra $\mathfrak{A}_{m,2}$. We characterize the sets of common eigenvalues of the Jucys–Murphy of the group ring of $G(m,1,n)$ and establish then the relation with the m-partitions. The representations of the group $G(m,1,n)$ are constructed again with the help of the tensor product of the algebra $\mathbb{C}G(m,1,n)$ with a free associative algebra generated by the standard m-tableaux corresponding to m-partitions of n. We also give in the first appendix to this section a proof of a structure theorem (giving a normal form) for the degenerate cyclotomic affine Hecke algebra (this is a particular case of the PBW basis of the wreath Hecke algebra given in [39]). In the second appendix we study the intertwining operators (introduced in [39]) in the degenerate cyclotomic affine Hecke algebra which provide a certain information about the spectrum of the Jucys–Murphy elements and explain how to obtain these intertwining operators by taking the classical limit of certain intertwining operators of the non-degenerate affine Hecke algebra.

Some material in this Section is known (the Jucys-Murphy elements [29, 40], the construction of the representations from the study of their spectrum [29], the degenerate cyclotomic affine Hecke algebra [39, 38]). We insist here on the connection of the treatment for the groups $G(m,1,n)$ with the treatment for the algebra $H(m,1,n)$. Also the construction of the representations given in Subsection 6.8 appears to be new.

6.1 Complex reflection group $G(m,1,n)$

The group $G(m,1,n)$ is generated by the elements t, s_1, …, s_{n-1} with the relations:

$$\begin{cases} s_i^2 = 1 & \text{for all } i = 1,\dots,n-1 \ , \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for all } i = 1,\dots,n-2 \ , \\ s_i s_j = s_j s_i & \text{for all } i,j = 1,\dots,n-1 \text{ such that } |i-j| > 1 \end{cases} \tag{6.2}$$

and

$$\begin{cases} t^m = 1 \ , \\ t s_1 t s_1 = s_1 t s_1 t \ , \\ t s_i = s_i t & \text{for } i > 1 \ . \end{cases} \tag{6.3}$$

The notation "s_i", the same as for the generators of the symmetric group S_n, see (4.4)-(4.6), should not lead to any confusion; the subgroup of the group $G(m,1,n)$ generated by the elements s_1, …, s_{n-1} is isomorphic to the symmetric group S_n.

The group $G(m,1,n)$ is isomorphic to the group $C_m \wr S_n$, the wreath product of the cyclic group with m elements, C_m, by the symmetric group S_n. Its order is $n! \cdot m^n$ (see Appendix A for a normal form). Let γ be a generator of the group C_m, that is $\gamma^m = e$ where e is the unit element of C_m. We also denote by e the unit element of S_n. The standard isomorphism between the group $G(m,1,n)$ and $C_m \wr S_n$ is given by the map:

$$t \mapsto \left(\begin{pmatrix} \gamma \\ e \\ \vdots \\ e \end{pmatrix}, e \right) \ , \quad s_i \mapsto \left(\begin{pmatrix} e \\ e \\ \vdots \\ e \end{pmatrix}, (i, i+1) \right) \ , \tag{6.4}$$

where the vectors are elements of the Cartesian product of n copies of C_m.

The group $G(m,1,n)$ admits the following equivalent presentation. Let E be the set $\{(\gamma^k, a),\ k = 0,\dots,m-1,\ a = 1,\dots,n\}$. Define the following action of C_m on this set: $\gamma \cdot (\gamma^k, a) = (\gamma^{k+1}, a)$. Denote $Perm(E)$ the group of permutations of the set E. Then the group $G(m,1,n)$ is isomorphic to the subgroup of $Perm(E)$ consisting of elements $\pi \in Perm(E)$ such that:

$$\pi(\gamma^k, a) = \gamma^k \cdot \pi(e, a) \quad \text{for all } k = 1,\dots,m-1 \text{ and } a = 1,\dots,n \ . \tag{6.5}$$

Indeed it is easy to see that this is a subgroup of $Perm(E)$ and that its order is $n! \cdot m^n$. We denote it by $Perm^0(E)$. In order to specify an element π of $Perm^0(E)$ it is enough to give the images under π of the elements of the set $\{(e,a), a = 1,\dots,n\}$. Let ϕ be the map from $G(m,1,n)$ to $Perm^0(E)$ defined on the generators of $G(m,1,n)$ by:

$$\begin{aligned} &\phi(t)(e,a) = \begin{cases} (\gamma, a) & \text{if } a = 1 \ , \\ (e, a) & \text{for } a \neq 1 \ ; \end{cases} \\ &\phi(s_i)(e,a) = (e, s_i(a)) \quad \text{for } a = 1,\dots,n \text{ and } i = 1,\dots,n-1 \ . \end{aligned} \tag{6.6}$$

We can extend ϕ to a homomorphism (it is enough to check that the maps $\phi(t), \phi(s_1), \dots, \phi(s_{n-1})$ satisfy the defining relations of $G(m,1,n)$). Moreover this homomorphism is surjective; it is a consequence of the following fact: for any $j = 1, \dots, n-1$,

$$\phi(s_j)\dots\phi(s_1)\phi(t)\phi(s_1)\dots\phi(s_j)(e,a) = \begin{cases} (\gamma, a) & \text{if } a = j+1 \ , \\ (e,a) & \text{for } a \neq j+1 \ . \end{cases} \tag{6.7}$$

Finally, as $G(m,1,n)$ and $Perm^0(E)$ have the same order, the map ϕ is an isomorphism.

6.2 Jucys–Murphy elements

Our main concern in this subsection is to obtain a version of Jucys–Murphy elements of $G(m,1,n)$ from the Jucys–Murphy elements of the cyclotomic Hecke algebra $H(m,1,n)$, more precisely, from limits of certain expressions containing the Jucys–Murphy elements of $H(m,1,n)$ (a similar process was used in [30] for the Weyl groups). The Jucys–Murphy elements were defined in [29, 40] for the wreath product of any finite group A by the symmetric group. The Jucys–Murphy elements obtained by a limiting procedure coincide with those from [29, 40] if we choose A to be the cyclic group.

As in the non-degenerate situation, the usage of the Jucys–Murphy elements is the main tool in our construction of the representation theory. We briefly outline the content of the following subsections. Once the Jucys–Murphy elements of $G(m,1,n)$ are obtained we verify that they realize the degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$ (see the Definition 17 below). We verify, on the classical level, the commutativity of the double set $\{x_1, \tilde{x}_1, \dots, x_n, \tilde{x}_n\}$ of elements in the algebra $\mathfrak{A}_{m,n}$ (we do not include the commutativity of this set in the defining relations of $\mathfrak{A}_{m,n}$, contrary to the definition of the wreath Hecke algebra of [39]; as a corollary of the results here, the two algebras are in fact isomorphic). The representation theory for the algebra $\mathfrak{A}_{m,2}$ carries an important information about the recursive properties of the Jucys–Murphy elements. We present the list of irreducible representations with diagonalizable $x_1, \tilde{x}_1, x_2$ and $\tilde{x}_2$ of the algebra $\mathfrak{A}_{m,2}$, and then, almost without proofs, the analogues of the results of Sections 3, 4 and 5 (the proofs mainly go along the same lines as the proofs of the analogous statements of Sections 3, 4 and 5) in the classical setting.

In the following we identify the generators τ, σ_1, …, σ_{n-1} of $H(m,1,n)$ with respectively t, s_1 ,…, s_{n-1} as soon as we have taken the classical limit $v_i \to \xi_i$ for $i = 1, \dots, m$ and $q \to \pm 1$.

Jucys–Murphy elements. We define the following classical analogues of the Jucys–Murphy elements J_i:

$$j_i := \lim_{q\to 1} \lim_{v_i \to \xi_i} \big(J_i\big) \ , \tag{6.8}$$

and

$$\tilde{j}_i := \frac{1}{m} \lim_{q\to 1} \lim_{v_i \to \xi_i} \left(\frac{J_i^m - 1}{q - q^{-1}} \right) \ . \tag{6.9}$$

Attention: the order of taking limits here is important, we first take the limit with respect to the variables v_i and then with respect to q; it is maybe more instructive to write (6.9) in the form $\tilde{j}_i := \frac{1}{m} \lim\limits_{q\to 1} \frac{\lim_{v_i\to\xi_i}\left(J_i^m - 1\right)}{q-q^{-1}}$.

6.3 Degenerate cyclotomic affine Hecke algebra

The Jucys–Murphy elements of the cyclotomic Hecke algebra $H(m,1,n)$ are the images of the "universal" Jucys–Murphy elements of the affine Hecke algebra. Similarly, the elements j_i and $\tilde{j}_i$ are the images of certain elements of a "universal" degenerate cyclotomic affine Hecke algebra, which we introduce here.

Definition 17. *Let $\mathfrak{A}_{m,n}$ be the algebra generated by $\overline{s}_1,\dots,\overline{s}_{n-1}$ and two more generators, x_1 and $\tilde{x}_1$; the defining relations we introduce in three steps. First, there are defining relations, involving the generators $\overline{s}_1,\dots,\overline{s}_{n-1}$ only:*

$$\begin{cases} \overline{s}_i^2 = 1 \ , \\ \overline{s}_i\overline{s}_{i+1}\overline{s}_i = \overline{s}_{i+1}\overline{s}_i\overline{s}_{i+1} \ , \\ \overline{s}_i\overline{s}_j = \overline{s}_j\overline{s}_i \quad \text{if} \ \ |i-j|>1 \ ; \end{cases} \tag{6.10}$$

second, there are relations concerning the addition of the generator x_1:

$$\begin{cases} x_1^m = 1 \ , \\ x_1\overline{s}_1x_1\overline{s}_1 = \overline{s}_1x_1\overline{s}_1x_1 \ , \\ x_1\overline{s}_i = \overline{s}_ix_1 \quad \text{if} \ \ i>2 \ ; \end{cases} \tag{6.11}$$

the third group of relations concerns the addition of the last generator $\tilde{x}_1$:

$$\begin{cases} \tilde{x}_1(\overline{s}_1\tilde{x}_1\overline{s}_1 + \frac{1}{m}\sum\limits_{p=1}^{m} x_1^p\overline{s}_1x_1^{-p}) = (\overline{s}_1\tilde{x}_1\overline{s}_1 + \frac{1}{m}\sum\limits_{p=1}^{m} x_1^p\overline{s}_1x_1^{-p})\tilde{x}_1 \ , \\ \tilde{x}_1\overline{s}_i = \overline{s}_i\tilde{x}_1 \quad \text{if} \ \ i>2 \ , \\ \tilde{x}_1x_1 = x_1\tilde{x}_1 \ , \\ \tilde{x}_1\overline{s}_1x_1\overline{s}_1 = \overline{s}_1x_1\overline{s}_1\tilde{x}_1 \ . \end{cases} \tag{6.12}$$

We call the algebra $\mathfrak{A}_{m,n}$ the degenerate cyclotomic affine Hecke algebra.

Due to the relations (6.10)–(6.11) there is a homomorphism

$$\hat{\imath}\ :\ \mathbb{C}G(m,1,n)\to\mathfrak{A}_{m,n}\ ,\quad \hat{\imath}(s_i)=\overline{s}_i\ \text{ for }\ i=1,\dots,n-1\ ,\ \hat{\imath}(t)=x_1\ . \tag{6.13}$$

Let π be a map from the set of generators $\{\overline{s}_1,\dots,\overline{s}_{n-1},x_1,\tilde{x}_1\}$ to $\mathbb{C}G(m,1,n)$ defined by

$$\pi\ :\ \overline{s}_i\mapsto s_i\ \text{ for }\ i=1,\dots,n-1\ ,\ x_1\mapsto t\ ,\ \tilde{x}_1\mapsto 0\ . \tag{6.14}$$

Clearly, π extends to a homomorphism, which we denote by the same symbol π, from the algebra $\mathfrak{A}_{m,n}$ to $\mathbb{C}G(m,1,n)$ (the homomorphism π is well defined since the relations (6.12) are trivially satisfied when one sends $\tilde{x}_1$ to 0). Moreover, the composition $\pi\circ\hat{\imath}$ leaves the generators of $G(m,1,n)$ invariant and is therefore the identity homomorphism of the algebra $\mathbb{C}G(m,1,n)$; in particular, the map $\hat{\imath}$ is injective or, equivalently, the subalgebra of $\mathfrak{A}_{m,n}$ generated by the elements $\overline{s}_1,\dots,\overline{s}_{n-1}$ and x_1 is isomorphic to the algebra $\mathbb{C}G(m,1,n)$.

Define "higher" elements x_i and $\tilde{x}_i$ for $i=2,\dots,n$ by

$$x_{i+1}=\overline{s}_i x_i \overline{s}_i\ ,\quad i=1,\dots,n-1\ , \tag{6.15}$$

and

$$\tilde{x}_{i+1}=\overline{s}_i \tilde{x}_i \overline{s}_i+\frac{1}{m}\sum_{p=1}^{m} x_i^p \overline{s}_i x_i^{-p}\ ,\quad i=1,\dots,n-1\ . \tag{6.16}$$

The second relation in (6.11) can be rewritten as

$$x_1x_2=x_2x_1\ ; \tag{6.17}$$

the first and the fourth relations in (6.12) can be rewritten, respectively, as

$$\tilde{x}_1\tilde{x}_2=\tilde{x}_2\tilde{x}_1\ \text{ and }\ \tilde{x}_1x_2=x_2\tilde{x}_1\ . \tag{6.18}$$

Lemma 18. *We have*

$$\pi(x_i)=j_i\ \ \textit{and}\ \ \pi(\tilde{x}_i)=\tilde{j}_i\ . \tag{6.19}$$

Proof. We have to check that the elements j_i (respectively, $\tilde{j}_i$) verify the recurrent relations (6.15) (respectively, (6.16)) and the initial conditions $j_1=t$ (respectively, $\tilde{j}_1=0$).

It follows from (6.8) in a straightforward manner that $j_1=t$ and $j_{i+1}=s_ij_is_i$ so only the verification for the elements $\tilde{j}_i$ remains.

Due to (6.9) we have $\tilde{j}_1=0$. Then we calculate

$$\begin{aligned} J_{i+1}^m &= (\sigma_iJ_i\sigma_i)^m\\ &=\sigma_iJ_i\Big((1+(q-q^{-1})\sigma_i)J_i\Big)^{m-1}\sigma_i\\ &=\sigma_iJ_i^m\sigma_i+(q-q^{-1})\Big(\sigma_iJ_i\sigma_iJ_i^{m-1}\sigma_i+\sigma_iJ_i^2\sigma_iJ_i^{m-2}\sigma_i+\dots+\sigma_iJ_i^{m-1}\sigma_iJ_i\sigma_i\Big)+O((q-q^{-1})^2)\ .\end{aligned}$$

Thus,

$$\begin{aligned}\frac{J_{i+1}^m-1}{q-q^{-1}} &= \frac{\sigma_iJ_i^m\sigma_i-1}{q-q^{-1}}+\Big(\sigma_iJ_i\sigma_iJ_i^{m-1}\sigma_i+\sigma_iJ_i^2\sigma_iJ_i^{m-2}\sigma_i+\dots+\sigma_iJ_i^{m-1}\sigma_iJ_i\sigma_i\Big)+O(q-q^{-1})\\ &=\frac{\sigma_i(J_i^m-1)\sigma_i}{q-q^{-1}}+\sigma_i+\Big(\sigma_iJ_i\sigma_iJ_i^{m-1}\sigma_i+\sigma_iJ_i^2\sigma_iJ_i^{m-2}\sigma_i+\dots+\sigma_iJ_i^{m-1}\sigma_iJ_i\sigma_i\Big)+O(q-q^{-1})\ .\end{aligned}$$

Upon taking the limit and dividing by m we obtain:

$$\tilde{j}_{i+1}=s_i\tilde{j}_is_i+\frac{1}{m}(s_i+\sum_{p=1}^{m-1}s_ij_i^ps_ij_i^{m-p}s_i)\ .$$

Finally we use that:

$$s_i j_i^p s_i j_i^q s_i = s_i j_i^p j_{i+1}^q = s_i j_{i+1}^q j_i^p = j_i^q s_i j_i^p \ ,$$

and we replace s_i by $j_i^m s_i j_i^0$ because $j_i^m = 1$. □

Since the Jucys–Murphy elements commute in the algebra $H(m,1,n)$, it follows from the definitions (6.8) and (6.9) that the elements j_i, $i = 1,\dots,n$, and the elements $\tilde{j}_i$, $i = 1,\dots,n$, form together a commutative set. We did not include the commutativity of the corresponding set, formed by the elements x_i, $i = 1,\dots,n$, and the elements $\tilde{x}_i$, $i = 1,\dots,n$, in the defining relations for the algebra $\mathfrak{A}_{m,n}$: the commutativity of this set (and therefore, by the Lemma 18, of its image under the morphism π, that is, of the set formed by the elements j_i, $i = 1,\dots,n$, and $\tilde{j}_i$, $i = 1,\dots,n$) follows, as we shall now see, from the relations (6.10)–(6.12).

Proposition 19. *The relations (6.10)–(6.12) imply that:*

$$x_k x_l = x_l x_k,\ \ \tilde{x}_k \tilde{x}_l = \tilde{x}_l \tilde{x}_k \ \textit{and}\ x_k \tilde{x}_l = \tilde{x}_l x_k \quad \textit{for all}\ \ k,l = 1,\dots,n\ . \tag{6.20}$$

Proof. We start by:

Lemma 20. *The relations (6.10)–(6.12) imply that x_i and $\tilde{x}_i$ commutes with $\overline{s}_k$ for $k > i$ and $k < i-1$.*

Proof of the Lemma. It is well known that the relations (6.10)–(6.11) imply that x_i commutes with $\overline{s}_k$ for $k > i$ and $k < i-1$.

We use induction on i for the elements $\tilde{x}_i$. By definition the element $\tilde{x}_1$ commutes with the elements $\overline{s}_k$ for $k > 1$; the element

$$\tilde{x}_{i+1} \ \equiv \ \overline{s}_i \tilde{x}_i \overline{s}_i + \frac{1}{m}\sum_{p=1}^{m} x_i^p \overline{s}_i x_i^{m-p} \tag{6.21}$$

commutes, by the induction hypothesis, with the elements $\overline{s}_k$ for $k > i+1$ and $k < i-1$. It is left to check the commutation relation with $\overline{s}_{i-1}$. This check is non-trivial only if $(i-1) > 0$; then we further decompose the elements $\tilde{x}_i$ and x_i in the right hand side of (6.21):

$$\begin{aligned}
\tilde{x}_{i+1} \ &= \overline{s}_i \tilde{x}_i \overline{s}_i + \tfrac{1}{m}\sum_{p=1}^{m} x_i^p \overline{s}_i x_i^{m-p} \\
&= \overline{s}_i \overline{s}_{i-1} \tilde{x}_{i-1} \overline{s}_{i-1} \overline{s}_i + \tfrac{1}{m}\sum_{p=1}^{m} \overline{s}_i x_{i-1}^p \overline{s}_{i-1} x_{i-1}^{m-p} \overline{s}_i + \tfrac{1}{m}\sum_{p=1}^{m} \overline{s}_{i-1} x_{i-1}^p \overline{s}_{i-1} \overline{s}_i \overline{s}_{i-1} x_{i-1}^{m-p} \overline{s}_{i-1} \\
&= \overline{s}_i \overline{s}_{i-1} \tilde{x}_{i-1} \overline{s}_{i-1} \overline{s}_i + \tfrac{1}{m}\sum_{p=1}^{m} \Big(x_{i-1}^p \overline{s}_i \overline{s}_{i-1} \overline{s}_i x_{i-1}^{m-p} + \overline{s}_{i-1} x_{i-1}^p \overline{s}_i \overline{s}_{i-1} \overline{s}_i x_{i-1}^{m-p} \overline{s}_{i-1} \Big)\ .
\end{aligned}$$

For each p, the expression in the last sum has the form $\xi + \overline{s}_{i-1}\xi\overline{s}_{i-1}$ (for some ξ) and therefore commutes with $\overline{s}_{i-1}$. The first term commutes with $\overline{s}_{i-1}$ as well:

$$\overline{s}_i \overline{s}_{i-1} \tilde{x}_{i-1} \overline{s}_{i-1} \overline{s}_i \cdot \overline{s}_{i-1} = \overline{s}_i \overline{s}_{i-1} \tilde{x}_{i-1} \overline{s}_i \overline{s}_{i-1} \overline{s}_i = \overline{s}_i \overline{s}_{i-1} \overline{s}_i \tilde{x}_{i-1} \overline{s}_{i-1} \overline{s}_i = \overline{s}_{i-1} \cdot \overline{s}_i \overline{s}_{i-1} \tilde{x}_{i-1} \overline{s}_{i-1} \overline{s}_i\ .$$

We conclude that $\tilde{x}_{i+1}$ commutes with $\overline{s}_{i-1}$. $\square$

We return to the proof of the Proposition 19. The commutativity statement (6.20) we establish by induction again. The element $\tilde{x}_1$ commutes with x_1 by definition. Assuming that $x_1,\dots,x_i,\tilde{x}_1,\dots,\tilde{x}_i$ form a commutative set we have to prove that x_{i+1} and $\tilde{x}_{i+1}$ commute with $x_1,\dots,x_i,\tilde{x}_1,\dots,\tilde{x}_i$ and that x_{i+1} commutes with $\tilde{x}_{i+1}$ as well.

(i) Since $x_{i+1}=\overline{s}_i x_i \overline{s}_i$ and $\tilde{x}_{i+1}=\overline{s}_i\tilde{x}_i\overline{s}_i+\frac{1}{m}\sum\limits_{p=1}^{m} x_i^p\overline{s}_i x_i^{m-p}$ we find, by the induction hypothesis and by the Lemma 20, that x_{i+1} and $\tilde{x}_{i+1}$ commute with x_k and $\tilde{x}_k$ for $k<i$.

(ii) We now prove that the elements x_{i+1} and $\tilde{x}_{i+1}$ commute with the elements x_i and $\tilde{x}_i$.

If $i>1$, we write $x_i=\overline{s}_{i-1}x_{i-1}\overline{s}_{i-1}$ and $\tilde{x}_i=\overline{s}_{i-1}\tilde{x}_{i-1}\overline{s}_{i-1}+\frac{1}{m}\sum\limits_{p=1}^{m} x_{i-1}^p\overline{s}_{i-1}x_{i-1}^{m-p}$; by the Lemma 20 and the statement (i) of the proof, the elements x_{i+1} and $\tilde{x}_{i+1}$ commute with all elements entering the above decompositions of x_i and $\tilde{x}_i$ and therefore, with x_i and $\tilde{x}_i$.

For $i=1$, the element x_2 commutes with x_1 and $\tilde{x}_1$ and the element $\tilde{x}_2$ commutes with $\tilde{x}_1$ by definition. To finish the proof of the statement that x_{i+1} and $\tilde{x}_{i+1}$ commute with x_i and $\tilde{x}_i$ it is left to show that $x_1\tilde{x}_2=\tilde{x}_2x_1$. We calculate

$$\begin{aligned} x_1\tilde{x}_2 &= x_1\overline{s}_1\tilde{x}_1\overline{s}_1+\tfrac{1}{m}\sum_{p=1}^{m} x_1^{p+1}\overline{s}_1x_1^{-p} \quad \text{and} \quad & \tilde{x}_2x_1 &= \overline{s}_1\tilde{x}_1\overline{s}_1x_1+\tfrac{1}{m}\sum_{p=1}^{m} x_1^{p}\overline{s}_1x_1^{1-p}\\ &= \overline{s}_1x_2\tilde{x}_1\overline{s}_1+\tfrac{1}{m}\sum_{p=1}^{m} x_1^{p+1}x_2^{-p}\overline{s}_1 & &= \overline{s}_1\tilde{x}_1x_2\overline{s}_1+\tfrac{1}{m}\sum_{p=1}^{m} x_1^{p}x_2^{1-p}\overline{s}_1\ , \end{aligned}$$

and, as $x_2\tilde{x}_1=\tilde{x}_1x_2$, the difference between $x_1\tilde{x}_2$ and $\tilde{x}_2x_1$ is $\frac{1}{m}(x_1^{m+1}x_2^{-m}-x_1)\overline{s}_1$ which is 0 because $x_1^m=x_2^m=1$.

(iii) It remains to prove that the elements x_{i+1} and $\tilde{x}_{i+1}$ commute. Using the already proved commutativity relations, we calculate:

$$\begin{aligned} \tilde{x}_{i+1}x_{i+1} &= \overline{s}_i\tilde{x}_ix_i\overline{s}_i+\tfrac{1}{m}\sum_{p=1}^{m} x_i^p\overline{s}_ix_i^{m-p}x_{i+1} \quad \text{and} \quad & x_{i+1}\tilde{x}_{i+1} &= \overline{s}_ix_i\tilde{x}_i\overline{s}_i+\tfrac{1}{m}\sum_{p=1}^{m} x_{i+1}x_i^p\overline{s}_ix_i^{m-p}\\ &= \overline{s}_ix_i\tilde{x}_i\overline{s}_i+\tfrac{1}{m}\sum_{p=1}^{m} x_i^p\overline{s}_ix_{i+1}x_i^{m-p} & &= \overline{s}_ix_i\tilde{x}_i\overline{s}_i+\tfrac{1}{m}\sum_{p=1}^{m} x_i^px_{i+1}\overline{s}_ix_i^{m-p}\\ &= \overline{s}_ix_i\tilde{x}_i\overline{s}_i+\tfrac{1}{m}\sum_{p=2}^{m+1} x_i^p\overline{s}_ix_i^{m-p+1}\ , & &= \overline{s}_ix_i\tilde{x}_i\overline{s}_i+\tfrac{1}{m}\sum_{p=1}^{m} x_i^p\overline{s}_ix_i^{m-p+1}\ , \end{aligned}$$

so the difference between $\tilde{x}_{i+1}x_{i+1}$ and $x_{i+1}\tilde{x}_{i+1}$ is equal to $\frac{1}{m}(x_i^{m+1}\overline{s}_i-x_i\overline{s}_ix_i^m)$ which is 0 because $x_i^m=1$. $\square$

6.4 Representations of algebra $\mathfrak{A}_{m,2}$

As in the non-degenerate situation, the important step in the understanding of the spectrum of the Jucys–Murphy elements and in the construction of representations is the analysis of the representations of the smallest non-trivial degenerate cyclotomic affine Hecke algebra, the algebra $\mathfrak{A}_{m,2}$. Here we present the list of irreducible representations with diagonalizable $x_1, \tilde{x}_1, x_2$ and $\tilde{x}_2$ of the algebra $\mathfrak{A}_{m,2}$.

Consider the algebra $\mathfrak{A}_{m,2}$ generated by x, y, $\tilde{x}$, $\tilde{y}$ and s with the relations:

$$\left\{\begin{array}{l} xy = yx\ , \quad \tilde{x}\tilde{y} = \tilde{y}\tilde{x}\ , \quad x\tilde{x} = \tilde{x}x\ , \quad y\tilde{x} = \tilde{x}y\ , \\ y = sxs\ , \quad x^m = 1\ , \quad \tilde{y} = s\tilde{x}s + \frac{1}{m}\sum\limits_{p=1}^{m} x^p s x^{m-p}\ , \quad s^2 = 1\ . \end{array}\right. \tag{6.22}$$

For all $i = 1, \dots, n-1$, the subalgebra of $\mathbb{C}G(m,1,n)$ generated by j_i, j_{i+1}, $\tilde{j}_i$, $\tilde{j}_{i+1}$ and s_i is a quotient of the algebra $\mathfrak{A}_{m,2}$. For $m = 1$ the algebra $\mathfrak{A}_{m,2}$ reduces to the degenerate affine Hecke algebra studied in [28] for the representation theory of the symmetric groups S_n.

The four elements x, y, $\tilde{x}$ and $\tilde{y}$ pairwise commute, see the Proposition 19. As for $H(m,1,n)$, we investigate irreducible representations of the algebra $\mathfrak{A}_{m,2}$ with diagonalizable x, y, $\tilde{x}$ and $\tilde{y}$. Let e be a common eigenvector of x, y, $\tilde{x}$ and $\tilde{y}$ with eigenvalues a, b, $\tilde{a}$ and $\tilde{b}$, respectively,

$$x.e = ae\ , \quad y.e = be\ , \quad \tilde{x}.e = \tilde{a}e\ , \quad \tilde{y}.e = \tilde{b}e\ , \tag{6.23}$$

where $\mathfrak{x}.\mathfrak{v}$ stands for the action of the element $\mathfrak{x}$ of the algebra on the vector $\mathfrak{v}$ of the representation space. We have $a^m = b^m = 1$. Using $xs = sy$, $ys = sx$, $\tilde{x}s = s\tilde{y} - \frac{1}{m}\sum_{p=1}^{m} y^p x^{m-p}$ and $\tilde{y}s = s\tilde{x} + \frac{1}{m}\sum_{p=1}^{m} y^p x^{m-p}$, we find

$$\begin{array}{c} x.(s.e) = bs.e\ , \quad y.(s.e) = as.e\ , \\ \tilde{x}.(s.e) = \tilde{b}s.e - \left(\frac{1}{m}\sum_{p=1}^{m} b^p a^{m-p}\right)e\ , \quad \tilde{y}.(s.e) = \tilde{a}s.e + \left(\frac{1}{m}\sum_{p=1}^{m} b^p a^{m-p}\right)e\ . \end{array} \tag{6.24}$$

Thus the action of the generators closes on the linear span of e and $s(e)$ and the irreducible representations can be only one-dimensional or two-dimensional. We straightforwardly arrive at the complete list of irreducible representations of the algebra $\mathfrak{A}_{m,2}$ with diagonalizable x, y, $\tilde{x}$ and $\tilde{y}$.

- The vector $s.e$ is proportional to the vector e. Then $s.e = \epsilon e$, where $\epsilon^2 = 1$; the representations of this type are one-dimensional. The action of generators is given by

$$x \mapsto a\ , \quad y \mapsto a\ , \quad \tilde{x} \mapsto \tilde{a}\ , \quad \tilde{y} \mapsto \tilde{a} + \epsilon\ , \quad s \mapsto \epsilon\ , \tag{6.25}$$

  where $a^m = 1$ and $\epsilon^2 = 1$.

- The vectors e and $s.e$ span a two-dimensional space. If $a \neq b$ then $\sum_{p=1}^{m} b^p a^{m-p} = 0$. The irreducible representations here are two-dimensional. The matrices of the generators of the algebra $\mathfrak{A}_{m,2}$ are given by

$$\begin{array}{c} s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad x \mapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\ , \quad y \mapsto \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}\ , \\ \tilde{x} \mapsto \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{b} \end{pmatrix}\ , \quad \tilde{y} \mapsto \begin{pmatrix} \tilde{b} & 0 \\ 0 & \tilde{a} \end{pmatrix}\ , \end{array} \tag{6.26}$$

  where $a^m = b^m = 1$ and $a \neq b$.

- The vectors e and $s.e$ span a two-dimensional space and $a=b$. Then $\frac{1}{m}\sum_{p=1}^{m} b^p a^{m-p}=1$. By (6.23)–(6.24) the action of $\tilde{x}$ and $\tilde{y}$ is diagonalizable if and only if $\tilde{a}\neq\tilde{b}$. The representations are two-dimensional. The matrices of the generators of the algebra $\mathfrak{A}_{m,2}$ in the basis $\{e,e'\}$, where $e':=s.e+\frac{1}{\tilde{a}-\tilde{b}}\,e$ are given by

$$s\mapsto\left(\begin{array}{cc}\frac{1}{\tilde{b}-\tilde{a}} & 1-\frac{1}{(\tilde{b}-\tilde{a})^2}\\ 1 & -\frac{1}{\tilde{b}-\tilde{a}}\end{array}\right)\ ,\quad x\mapsto\left(\begin{array}{cc}a & 0\\ 0 & a\end{array}\right)\ ,\quad y\mapsto\left(\begin{array}{cc}a & 0\\ 0 & a\end{array}\right)\ ,$$
$$\tilde{x}\mapsto\left(\begin{array}{cc}\tilde{a} & 0\\ 0 & \tilde{b}\end{array}\right)\ ,\quad \tilde{y}\mapsto\left(\begin{array}{cc}\tilde{b} & 0\\ 0 & \tilde{a}\end{array}\right)\ . \tag{6.27}$$

where $a^m=1$ and $\tilde{b}\neq\tilde{a}$. The representation (6.27) is irreducible if and only if $\tilde{b}\neq\tilde{a}\pm 1$.

6.5 Classical spectrum

In this and the two following subsections, the classical analogues of the results of Section 3 are given without proofs (which mostly repeat the proofs of the analogous statements of Section 3).

As for $H(m,1,n)$, the first step consists in construction of all representations of $\mathbb{C}G(m,1,n)$ verifying two conditions. First, the classical Jucys–Murphy elements $j_1,\dots,j_n,\tilde{j}_1,\dots,\tilde{j}_n$ are represented by semi-simple (diagonalizable) operators. Second, for every $i=1,\dots,n-1$ the action of the subalgebra generated by j_i, j_{i+1}, $\tilde{j}_i$, $\tilde{j}_{i+1}$ and s_i is completely reducible. We shall keep the name C-representations for these representations. At the end of the construction we shall see that all irreducible representations of $\mathbb{C}G(m,1,n)$ are C-representations.

We denote Spec $\left(\begin{array}{ccc}j_1 & ,\dots, & j_n\\ \tilde{j}_1 & ,\dots, & \tilde{j}_n\end{array}\right)$ the set of common eigenvalues of the elements $j_1,\tilde{j}_1,\dots,j_n,\tilde{j}_n$ in the C-representations:

$$\Lambda=\left(\begin{array}{ccc}a_1^{(\Lambda)} & ,\dots, & a_n^{(\Lambda)}\\ \tilde{a}_1^{(\Lambda)} & ,\dots, & \tilde{a}_n^{(\Lambda)}\end{array}\right) \tag{6.28}$$

belongs to Spec $\left(\begin{array}{ccc}j_1 & ,\dots, & j_n\\ \tilde{j}_1 & ,\dots, & \tilde{j}_n\end{array}\right)$ if there is a vector e_Λ in the space of a C-representation such that $j_i(e_\Lambda)=a_i^{(\Lambda)}e_\Lambda$ and $\tilde{j}_i(e_\Lambda)=\tilde{a}_i^{(\Lambda)}e_\Lambda$ for all $i=1,\dots,n$.

The elements j_i and $\tilde{j}_i$ commute with s_k for $k>i$ and $k<i-1$ (see the Lemma 20) which implies that the action of s_k on Spec $\left(\begin{array}{ccc}j_1 & ,\dots, & j_n\\ \tilde{j}_1 & ,\dots, & \tilde{j}_n\end{array}\right)$ is "local" in the sense that $s_k(e_\Lambda)$ is a linear combination of $e_{\Lambda'}$ such that $a_i^{(\Lambda')}=a_i^{(\Lambda)}$ and $\tilde{a}_i^{(\Lambda')}=\tilde{a}_i^{(\Lambda)}$ for $i\neq k,k+1$.

The $2\times n$ arrays (6.28) we shall call strings, keeping the name "string" used for a set of common eigenvalues of the Jucys–Murphy elements for the algebra $H(m,1,n)$.

Proposition 21. *Let* $\Lambda = \begin{pmatrix} a_1 & ,\dots, & a_i \ , \ a_{i+1} & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_i \ , \ \tilde{a}_{i+1} & ,\dots, & \tilde{a}_n \end{pmatrix} \in \operatorname{Spec}\begin{pmatrix} j_1 & ,\dots, & j_n \\ \tilde{j}_1 & ,\dots, & \tilde{j}_n \end{pmatrix}$ *and let* e_Λ *be a corresponding vector. Then*

(a) We have $a_i^m = 1$ *for all* $i = 1, \dots, n$*; if* $a_i = a_{i+1}$ *then* $\tilde{a}_i \neq \tilde{a}_{i+1}$.

(b) If $a_{i+1} = a_i$ *and* $\tilde{a}_{i+1} = \tilde{a}_i + \epsilon$*, where* $\epsilon = \pm 1$*, then* $s_i(e_\Lambda) = \epsilon e_\Lambda$.

(c) If $a_{i+1} \neq a_i$ *or* $a_{i+1} = a_i$ & $\tilde{a}_{i+1} \neq \tilde{a}_i \pm 1$ *then*

$$\Lambda' = \begin{pmatrix} a_1 & ,\dots, & a_{i+1} \ , \ a_i & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_{i+1} \ , \ \tilde{a}_i & ,\dots, & \tilde{a}_n \end{pmatrix} \in \operatorname{Spec}\begin{pmatrix} j_1 & ,\dots, & j_n \\ \tilde{j}_1 & ,\dots, & \tilde{j}_n \end{pmatrix} .$$

Moreover, if $a_{i+1} \neq a_i$ *then the vector* $s_i(e_\Lambda)$ *corresponds to the string* Λ'*, see the matrices (6.26) with* $a = a_i$*,* $b = a_{i+1}$*,* $\tilde{a} = \tilde{a}_i$ *and* $\tilde{b} = \tilde{a}_{i+1}$*; if* $a_{i+1} = a_i$ *and* $\tilde{a}_{i+1} \neq \tilde{a}_i \pm 1$ *then the vector* $s_i(e_\Lambda) - \frac{1}{\tilde{a}_{i+1} - \tilde{a}_i} e_\Lambda$ *corresponds to the string* Λ'*, see the matrices (6.27) with* $a = a_i = a_{i+1}$*,* $\tilde{a} = \tilde{a}_i$ *and* $\tilde{b} = \tilde{a}_{i+1}$.

6.6 Classical content strings

We define the classical analogue of the set $\mathrm{Cont}_m(n)$ which we denote by $\mathrm{cCont}_m(n)$.

Recall that $[k, l] = \{k, k+1, \dots, l-1, l\}$ for two integers $k, l \in \mathbb{Z}$, $k < l$.

Definition 22. *A classical content string* $\begin{pmatrix} a_1 & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_n \end{pmatrix}$ *is a string of columns of numbers satisfying the following conditions:*

(1) $\tilde{a}_1 = 0$ *and* $a_i^m = 1$ *for all* $i = 1, \dots, n$.

(2) For all $j > 1$*: if* $\tilde{a}_j \neq 0$ *then there exists* i*,* $i < j$*, such that* $a_i = a_j$ *and* $\tilde{a}_i \in \{\tilde{a}_j - 1, \tilde{a}_j + 1\}$.

(3) If $a_j = a_k$ *and* $\tilde{a}_j = \tilde{a}_k$ *for* j, k*,* $j < k$*, then there exist* $i_1, i_2 \in [j+1, k-1]$ *such that* $a_{i_1} = a_{i_2} = a_j = a_k$*,* $\tilde{a}_{i_1} = \tilde{a}_j - 1$ *and* $\tilde{a}_{i_2} = \tilde{a}_j + 1$.

The set of classical content strings we denote by $\mathrm{cCont}_m(n)$.

Here is the classical analogue of the Proposition 3.

Proposition 23. *Assume that a string of columns of numbers* $\begin{pmatrix} a_1 & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_n \end{pmatrix}$ *belongs to the set* $\operatorname{Spec}\begin{pmatrix} j_1 & ,\dots, & j_n \\ \tilde{j}_1 & ,\dots, & \tilde{j}_n \end{pmatrix}$. *Then it belongs to the set* $\mathrm{cCont}_m(n)$.

Remark. It follows directly from the Definition 22 that if

$$\left(\begin{array}{ccccccc} a_1 & ,\dots, & a_i & , & a_{i+1} & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_i & , & \tilde{a}_{i+1} & ,\dots, & \tilde{a}_n \end{array}\right) \in \mathrm{cCont}_m(n)$$

with $a_{i+1} \neq a_i$ or with $a_{i+1} = a_i$ & $\tilde{a}_{i+1} \neq \tilde{a}_i \pm 1$ then

$$\left(\begin{array}{ccccccc} a_1 & ,\dots, & a_{i+1} & , & a_i & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_{i+1} & , & \tilde{a}_i & ,\dots, & \tilde{a}_n \end{array}\right) \in \mathrm{cCont}_m(n).$$

Like in the remark just after the proof of the Proposition 3, the action (described in the Proposition 21) of the generators s_i, $i = 1, \dots, n-1$, on the vector space with a basis formed by vectors e_Λ, $\Lambda \in \mathrm{Spec}\left(\begin{array}{ccc} j_1 & ,\dots, & j_n \\ \tilde{j}_1 & ,\dots, & \tilde{j}_n \end{array}\right)$, extends to an action on the vector space with a basis formed by vectors e_μ, $\mu \in \mathrm{cCont}_m(n)$. The classical analogues of other statements in the remark after the proof of the Proposition 3 hold as well.

6.7 Classical content of an m-node in a Young m-diagram

Proposition 24. *There is a bijection between the set of standard Young m-tableaux of length n and the set* $\mathrm{cCont}_m(n)$.

This classical analogue of the Proposition 5 is proved along the same lines as the Proposition 5; we need only to modify the notion of a content of an m-node in a Young m-diagram.

The classical content of a node in a Young diagram is $(s-r)$ when the node lies in the line r and column s. To extend this definition to Young m-diagrams, we have to specify in which diagram of an m-diagram the m-node lies; thus, the content of an m-node in a Young m-diagram is a couple of numbers, the first number specifies the diagram (in which the m-node lies) in the m-diagram and the second number gives the content of the m-node in the specified diagram. To relate this information with the spectra of the Jucys–Murphy elements, fix (arbitrarily) a bijection between the set $[1, m]$ and the set of distinct m-th roots of unity; let ξ_k be the root of unity associated with $k \in [1, m]$ by this bijection. We define the classical content of an m-node which lies in the line r and column s of the k^{th} diagram of the m-diagram to be the column $\left(\begin{array}{c} \xi_k \\ s-r \end{array}\right)$.

Now to a standard Young m-tableau with length n we associate a string of columns of numbers $\left(\begin{array}{ccc} a_1 & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_n \end{array}\right)$ where $\left(\begin{array}{c} a_i \\ \tilde{a}_i \end{array}\right)$ is the content of the m-node in which the number i is placed in the m-tableau. This association provides, like in the proof of the Proposition 5, the bijection stated in the Proposition 24.

Here is the same example as in paragraph 6, Section 3, but in the classical context:

$$\left(\begin{array}{|c|c|c|} \hline {}^{1}\,0 & {}^{2}\,1 & {}^{4}\,2 \\ \hline {}^{6}\,{-1} & {}^{9}\,0 & \multicolumn{1}{c}{} \\ \cline{1-2} {}^{7}\,{-2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|} \hline {}^{3}\,0 & {}^{8}\,1 & {}^{10}\,2 \\ \hline {}^{5}\,{-1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) \tag{6.29}$$

The string, associated to this standard Young 2-tableau, is:

$$\left(\begin{array}{ccccccccccccccccccc} \xi_1 & , & \xi_1 & , & \xi_2 & , & \xi_1 & , & \xi_2 & , & \xi_1 & , & \xi_1 & , & \xi_2 & , & \xi_1 & , & \xi_2 \\ 0 & , & 1 & , & 0 & , & 2 & , & -1 & , & -1 & , & -2 & , & 1 & , & 0 & , & 2 \end{array} \right) ,$$

where $\{\xi_1, \xi_2\}$ is the set of distinct square roots of unity.

Remark. In the classical limit the elements j_i serve the same aim in the representation theory as the numbers v_k in front of the powers of q in the spectrum of the non-degenerate elements J_i: the elements j_i distinguish different Young tableaux in an m-tableau.

6.8 Construction of representations

Here we establish an analogue of the construction from Section 4 in the classical setting: we define an algebra structure on a tensor product of the algebra $\mathbb{C}G(m,1,n)$ with a free associative algebra generated by the standard m-tableaux corresponding to m-partitions of n. Then, by evaluation (with the help of the simplest one-dimensional representation of $G(m,1,n)$) from the right, we build representations. We do not give the proofs of the statements when they are completely similar to the proofs of the analogous statements from Section 4; we only indicate the modifications.

6.8.1 Baxterized elements

Define, for any s_i among the generators $s_1, \dots, s_{n-1}$ of $G(m,1,n)$, the Baxterized element $s_i(\alpha,\beta)$ by

$$s_i(\alpha,\beta) := s_i + \frac{1}{\alpha-\beta} \ . \tag{6.30}$$

The parameters α and β are called spectral parameters.

Proposition 25. *The following relations hold:*

$$\begin{array}{rcll} s_i(\alpha,\beta)s_i(\beta,\alpha) & = & 1 - \frac{1}{(\alpha-\beta)^2} \ , & \\ s_i(\alpha,\beta)s_{i+1}(\alpha,\gamma)s_i(\beta,\gamma) & = & s_{i+1}(\beta,\gamma)s_i(\alpha,\gamma)s_{i+1}(\alpha,\beta) \ , & \\ s_i(\alpha,\beta)s_j(\gamma,\delta) & = & s_j(\gamma,\delta)s_i(\alpha,\beta) & \text{if } |i-j|>1 \ . \end{array} \tag{6.31}$$

As in the non-classical situation the original relations follow from the relations for the Baxterized elements with fixed values of the spectral parameters.

Lemma 26. *Let A and B be two elements of an arbitrary associative unital algebra $\mathcal{A}$. Denote $A(\alpha,\beta) := A + \frac{1}{\alpha-\beta}$ and $B(\alpha,\beta) := B + \frac{1}{\alpha-\beta}$ where α and β are parameters.*

(i) If

$$A(\alpha,\beta)A(\beta,\alpha) = 1 - \frac{1}{(\alpha-\beta)^2} \ ,$$

for some (arbitrarily) fixed values of the parameters α and β ($\alpha \neq \beta$) then

$$A^2 = 1 \ .$$

(ii) If $A^2 = 1$, $B^2 = 1$ and

$$A(\alpha,\beta)B(\alpha,\gamma)A(\beta,\gamma) = B(\beta,\gamma)A(\alpha,\gamma)B(\alpha,\beta)$$

for some (arbitrarily) fixed values of the parameters α,β and γ ($\alpha \neq \beta \neq \gamma \neq \alpha$) then

$$ABA = BAB \ .$$

(iii) If

$$A(\alpha,\beta)B(\gamma,\delta) = B(\gamma,\delta)A(\alpha,\beta)$$

for some (arbitrarily) fixed values of the parameters α,β,γ and δ ($\alpha \neq \beta$ and $\gamma \neq \delta$) then

$$AB = BA \ .$$

6.8.2 Product of the algebra $\mathbb{C}G(m,1,n)$ with a free associative algebra generated by the standard m-tableaux corresponding to m-partitions of n

Let $\lambda^{(m)}$ be an m-partition of length n. Consider a set of free generators labeled by standard m-tableaux of shape $\lambda^{(m)}$; for a standard m-tableau $X_{\lambda^{(m)}}$ we denote, as before, by $\mathcal{X}_{\lambda^{(m)}}$ the corresponding free generator.

Recall that the classical content of an m-node which lies in the line r and column s of the k^{th} diagram of the m-diagram is the column $\begin{pmatrix} \xi_k \\ s-r \end{pmatrix}$, where $k \mapsto \xi_k$ is an arbitrarily chosen bijection between the set $[1,m]$ and the set of distinct m-th roots of unity. For a standard m-tableau we shall denote the entries of the content column of an m-node where i is placed by $\begin{pmatrix} p(X_{\lambda^{(m)}}|i) \\ cc(X_{\lambda^{(m)}}|i) \end{pmatrix}$.

Proposition 27. *The following relations:*

- *if $p(X_{\lambda^{(m)}}|i) \neq p(X_{\lambda^{(m)}}|i+1)$ then*

$$s_i \cdot \mathcal{X}_{\lambda^{(m)}} = \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot s_i \ ; \tag{6.32}$$

- *if* $p(X_{\lambda^{(m)}}|i) = p(X_{\lambda^{(m)}}|i+1)$ *then*

$$\Big(s_i + \frac{1}{cc(X_{\lambda^{(m)}}|i) - cc(X_{\lambda^{(m)}}|i+1)}\Big) \cdot \mathcal{X}_{\lambda^{(m)}} = \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \Big(s_i + \frac{1}{cc(X_{\lambda^{(m)}}|i+1) - cc(X_{\lambda^{(m)}}|i)}\Big) \tag{6.33}$$

and

$$\Big(t - p(X_{\lambda^{(m)}}|1)\Big) \cdot \mathcal{X}_{\lambda^{(m)}} = 0 \tag{6.34}$$

are compatible with the relations for the generators $t, s_1, \dots, s_{n-1}$ *of the group* $G(m,1,n)$*. The element* $\mathcal{X}^{s_i}_{\lambda^{(m)}}$ *corresponds to the m-tableau obtained from* $X_{\lambda^{(m)}}$ *by exchanging the m-nodes with numbers i and* $(i+1)$*. If the resulting m-tableau is not standard, we put* $\mathcal{X}^{s_i}_{\lambda^{(m)}} = 0$*.*

The compatibility is understood in the same sense as in the explanations after the formulation of the Proposition 8. We denote the resulting algebra by $\mathfrak{T}_c$.

Proof. Notice that for $p(X_{\lambda^{(m)}}|i) = p(X_{\lambda^{(m)}}|i+1)$ the relation (6.33) can be rewritten with the use of the Baxterized form of the elements s_i:

$$s_i\big(cc(X_{\lambda^{(m)}}|i), cc(X_{\lambda^{(m)}}|i+1)\big) \cdot \mathcal{X}_{\lambda^{(m)}} = \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot s_i\big(cc(X_{\lambda^{(m)}}|i+1), cc(X_{\lambda^{(m)}}|i)\big) \ .$$

(i) If $p(X_{\lambda^{(m)}}|i) = p(X_{\lambda^{(m)}}|i+1)$ we prove the compatibility of the relation $s_i^2 = 1$ with the instructions (6.32)–(6.34) by a calculation similar to the one from the proof of the Proposition 8; one uses here the Proposition 25 and the Lemma 26 instead of the Proposition 6 and the Lemma 7.

If $p(X_{\lambda^{(m)}}|i) \neq p(X_{\lambda^{(m)}}|i+1)$ the compatibility is immediate.

(ii) If $p(X_{\lambda^{(m)}}|i) = p(X_{\lambda^{(m)}}|i+1) = p(X_{\lambda^{(m)}}|i+2)$ we prove the compatibility of the relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ with the instructions (6.32)–(6.34) by a calculation similar to the one from the proof of the Proposition 8 (with the use of the Proposition 25 and the Lemma 26 instead of the Proposition 6 and the Lemma 7).

The case $p(X_{\lambda^{(m)}}|i) \neq p(X_{\lambda^{(m)}}|i+1) \neq p(X_{\lambda^{(m)}}|i+2) \neq p(X_{\lambda^{(m)}}|i)$ is immediate.

Three cases remain:

- $p(X_{\lambda^{(m)}}|i) \neq p(X_{\lambda^{(m)}}|i+1)$ and $p(X_{\lambda^{(m)}}|i+1) = p(X_{\lambda^{(m)}}|i+2)$,
- $p(X_{\lambda^{(m)}}|i) = p(X_{\lambda^{(m)}}|i+1)$ and $p(X_{\lambda^{(m)}}|i+1) \neq p(X_{\lambda^{(m)}}|i+2)$,
- $p(X_{\lambda^{(m)}}|i) = p(X_{\lambda^{(m)}}|i+2)$ and $p(X_{\lambda^{(m)}}|i+1) \neq p(X_{\lambda^{(m)}}|i+2)$.

In each of these cases we perform a straightforward calculation using that, by definition, for any permutation $\pi \in S_n$,

$$cc(X^{\pi}_{\lambda^{(m)}}|k) = cc(X_{\lambda^{(m)}}|\pi^{-1}(k)) \text{ and } p(X^{\pi}_{\lambda^{(m)}}|k) = p(X_{\lambda^{(m)}}|\pi^{-1}(k)) \ .$$

We write down this calculation only for the case $p(X_{\lambda^{(m)}}|i) \neq p(X_{\lambda^{(m)}}|i+1)$ and $p(X_{\lambda^{(m)}}|i+1) = p(X_{\lambda^{(m)}}|i+2)$ (the two other calculations are very similar). For brevity we denote $cc^{(k)} = cc(X_{\lambda^{(m)}}|k)$.

We have

$$\begin{aligned}
s_is_{i+1}s_i\cdot\mathcal{X}_{\lambda^{(m)}} &= s_is_{i+1}\cdot\mathcal{X}^{s_i}_{\lambda^{(m)}}\cdot s_i\\
&= s_i\cdot\mathcal{X}^{s_{i+1}s_i}_{\lambda^{(m)}}\cdot s_{i+1}s_i\\
&= -\tfrac{1}{cc^{(i+1)}-cc^{(i+2)}}\mathcal{X}^{s_{i+1}s_i}_{\lambda^{(m)}}\cdot s_{i+1}s_i+\mathcal{X}^{s_is_{i+1}s_i}_{\lambda^{(m)}}\cdot(s_i+\tfrac{1}{cc^{(i+2)}-cc^{(i+1)}})s_{i+1}s_i\ ,
\end{aligned}$$

and

$$\begin{aligned}
s_{i+1}s_is_{i+1}\cdot\mathcal{X}_{\lambda^{(m)}} &= s_{i+1}s_i\cdot\big(-\tfrac{1}{cc^{(i+1)}-cc^{(i+2)}}\mathcal{X}_{\lambda^{(m)}}+\mathcal{X}^{s_{i+1}}_{\lambda^{(m)}}\cdot(s_{i+1}+\tfrac{1}{cc^{(i+2)}-cc^{(i+1)}})\big)\\
&= s_{i+1}\cdot\big(-\tfrac{1}{cc^{(i+1)}-cc^{(i+2)}}\mathcal{X}^{s_i}_{\lambda^{(m)}}\cdot s_i+\mathcal{X}^{s_is_{i+1}}_{\lambda^{(m)}}\cdot s_i(s_{i+1}+\tfrac{1}{cc^{(i+2)}-cc^{(i+1)}})\big)\\
&= -\tfrac{1}{cc^{(i+1)}-cc^{(i+2)}}\mathcal{X}^{s_{i+1}s_i}_{\lambda^{(m)}}\cdot s_{i+1}s_i+\mathcal{X}^{s_{i+1}s_is_{i+1}}_{\lambda^{(m)}}\cdot s_{i+1}s_i(s_{i+1}+\tfrac{1}{cc^{(i+2)}-cc^{(i+1)}})\ .
\end{aligned}$$

Thus $(s_is_{i+1}s_i-s_{i+1}s_is_{i+1})\cdot\mathcal{X}_{\lambda^{(m)}}=\mathcal{X}^{s_is_{i+1}s_i}_{\lambda^{(m)}}\cdot(s_is_{i+1}s_i-s_{i+1}s_is_{i+1})$ and so the compatibility of the relation $s_is_{i+1}s_i=s_{i+1}s_is_{i+1}$ with the instructions (6.32)–(6.34) is proven.

(iii) If $p(X_{\lambda^{(m)}}|i)=p(X_{\lambda^{(m)}}|i+1)$ and $p(X_{\lambda^{(m)}}|j)=p(X_{\lambda^{(m)}}|j+1)$ we prove the compatibility of the relation $s_is_j=s_js_i$ with the instructions (6.32)–(6.34) by a calculation similar to the one from the proof of the Proposition 8 (with the use of the Proposition 25 and the Lemma 26 instead of the Proposition 6 and the Lemma 7).

If $p(X_{\lambda^{(m)}}|i)\neq p(X_{\lambda^{(m)}}|i+1)$ or $p(X_{\lambda^{(m)}}|j)\neq p(X_{\lambda^{(m)}}|j+1)$ the compatibility follows from an easy calculation.

The verification of the compatibility of the relations $t^m=1$ and $ts_i=s_it$ for $i>1$ with the instructions (6.32)–(6.34) is immediate.

The compatibility of the relation $ts_1ts_1=s_1ts_1t$ with the instructions (6.32)–(6.34) is a direct consequence of the Lemma below. □

Lemma 28. *The relations (6.32) imply the relations:*

$$\big(j_i-p(X_{\lambda^{(m)}}|i)\big)\cdot\mathcal{X}_{\lambda^{(m)}}=0\quad \textit{for all } i=1,\dots,n\ , \tag{6.35}$$

$$\big(\tilde{j}_i-cc(X_{\lambda^{(m)}}|i)\big)\cdot\mathcal{X}_{\lambda^{(m)}}=0\quad \textit{for all } i=1,\dots,n\ . \tag{6.36}$$

Proof. Let $X_{\lambda^{(m)}}$ be a standard m-tableau and set, for brevity, $p^{(i)}=p(X_{\lambda^{(m)}}|i)$ and $cc^{(i)}=cc(X_{\lambda^{(m)}}|i)$ for all $i=1,\dots,n$. We prove (6.35)-(6.36) by induction on i. The basis of induction for (6.35) is the relation (6.34) and the basis of induction for (6.36) is trivial since $\tilde{j}_1=0$ and $cc^{(1)}=0$.

If $p^{(i+1)}=p^{(i)}$ then the proof of (6.35) is very similar to the proof of the Lemma 9. If $p^{(i+1)}\neq p^{(i)}$ then (6.35) is immediate.

Now we prove (6.36). We recall that $\tilde{j}_{i+1}=s_i\tilde{j}_is_i+\frac{1}{m}\sum\limits_{k=1}^{m}j_i^ks_ij_i^{-k}$.

Assume first that $X^{s_i}_{\lambda^{(m)}}$ is not standard. Then $p^{(i+1)} = p^{(i)}$, $cc^{(i+1)} = cc^{(i)} + \epsilon$ and $s_i \cdot \mathcal{X}_{\lambda^{(m)}} = \epsilon \mathcal{X}_{\lambda^{(m)}}$ with $\epsilon = \pm 1$. It is immediate that $\tilde{j}_{i+1} \cdot \mathcal{X}_{\lambda^{(m)}} = cc^{(i+1)} \mathcal{X}_{\lambda^{(m)}}$.

Next assume that $X^{s_i}_{\lambda^{(m)}}$ is standard and that $p^{(i+1)} \neq p^{(i)}$. Then $s_i \cdot \mathcal{X}_{\lambda^{(m)}} = \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot s_i$ and we obtain

$$\tilde{j}_{i+1} \cdot \mathcal{X}_{\lambda^{(m)}} = cc^{(i+1)} \mathcal{X}_{\lambda^{(m)}} + \frac{1}{m} \sum_{k=1}^{m} (p^{(i)})^{-k} (p^{(i+1)})^{k} \mathcal{X}^{s_i}_{\lambda^{(m)}} \ .$$

As $\sum\limits_{k=1}^{m} (p^{(i)})^{-k} (p^{(i+1)})^{k} = 0$ (since $p^{(i+1)} \neq p^{(i)}$), we have $\tilde{j}_{i+1} \cdot \mathcal{X}_{\lambda^{(m)}} = cc^{(i+1)} \mathcal{X}_{\lambda^{(m)}}$.

Finally assume that $X^{s_i}_{\lambda^{(m)}}$ is standard and that $p^{(i+1)} = p^{(i)}$. Straightforward calculations lead to

$$s_i \tilde{j}_i s_i \cdot \mathcal{X}_{\lambda^{(m)}} = \left(cc^{(i+1)} + \frac{1}{cc^{(i)} - cc^{(i+1)}} \right) \mathcal{X}_{\lambda^{(m)}} - \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \left(s_i + \frac{1}{cc^{(i+1)} - cc^{(i)}} \right) \ ,$$

and

$$\left(\frac{1}{m} \sum_{k=1}^{m} j_i^k s_i j_i^{-k} \right) \cdot \mathcal{X}_{\lambda^{(m)}} = \frac{-1}{cc^{(i)} - cc^{(i+1)}} \mathcal{X}_{\lambda^{(m)}} + \mathcal{X}^{s_i}_{\lambda^{(m)}} \cdot \left(s_i + \frac{1}{cc^{(i+1)} - cc^{(i)}} \right) \ .$$

Adding these two equalities, we obtain $\tilde{j}_{i+1} \cdot \mathcal{X}_{\lambda^{(m)}} = cc^{(i+1)} \mathcal{X}_{\lambda^{(m)}}$. $\square$

6.8.3 Representations

We apply the same procedure as at the end of Section 4 to build representations of $G(m,1,n)$ on the vector space $U_{\lambda^{(m)}}$ spanned by $\{\mathcal{X}_{\lambda^{(m)}}\}$. We use as the "vacuum" $|\rangle$ the basic vector of the one-dimensional $G(m,1,n)$-module: $s_i |\rangle = |\rangle$ and $t|\rangle = \xi_1 |\rangle$. This procedure leads to the following formulas for the action of the generators $t, s_1, \dots, s_{n-1}$ on the basis vectors $\mathcal{X}_{\lambda^{(m)}}$ of $U_{\lambda^{(m)}}$:

- if $p(X_{\lambda^{(m)}}|i) \neq p(X_{\lambda^{(m)}}|i+1)$ then

$$s_i \ : \ \mathcal{X}_{\lambda^{(m)}} \mapsto \mathcal{X}^{s_i}_{\lambda^{(m)}} \ , \tag{6.37}$$

- if $p(X_{\lambda^{(m)}}|i) = p(X_{\lambda^{(m)}}|i+1)$ then

$$\begin{aligned} s_i \ : \ \mathcal{X}_{\lambda^{(m)}} \ \mapsto \ & -\frac{1}{cc(X_{\lambda^{(m)}}|i) - cc(X_{\lambda^{(m)}}|i+1)} \mathcal{X}_{\lambda^{(m)}} \\ & + \left(1 + \frac{1}{cc(X_{\lambda^{(m)}}|i+1) - cc(X_{\lambda^{(m)}}|i)} \right) \mathcal{X}^{s_i}_{\lambda^{(m)}} \ , \end{aligned} \tag{6.38}$$

and

$$t \ : \ \mathcal{X}_{\lambda^{(m)}} \mapsto p(X_{\lambda^{(m)}}|1) \mathcal{X}_{\lambda^{(m)}} \ . \tag{6.39}$$

As before, it is assumed here that $\mathcal{X}^{s_i}_{\lambda^{(m)}} = 0$ if $X^{s_i}_{\lambda^{(m)}}$ is not a standard m-tableau.

Remarks.

(a) As for the algebra $H(m,1,n)$ (see remarks at the end of Subsection 4.3), certain properties of the action of the generator t repeat corresponding properties of the generator s_1 in the representation theory of the usual Hecke algebra.

Moreover, again as for the cyclotomic Hecke algebra $H(m,1,n)$, the constructed representations do not depend (up to isomorphism) on the value of the generators $s_1,\dots,s_{n-1}$ and t on the vacuum $|\rangle$.

(b) In the same way as for the cyclotomic Hecke algebras, the associative algebra structure on the tensor product of the algebra $\mathbb{C}G(m,1,n)$ with a free associative algebra generated by the standard m-tableaux allows to equip the tensor products of the spaces of representations corresponding to two (in general, any number of) m-diagrams with the $\mathbb{C}G(m,1,n)$-module structure. The decomposition rules of the tensor structure are given by the same formula (4.33).

(c) This remark is the analogue of the remark **(e)** at the end of Subsection 4.3 in the classical situation. In Appendix 6.B we introduce the classical intertwining operators $\tilde{u}_{i+1} := \overline{s}_i\tilde{x}_i - \tilde{x}_i\overline{s}_i \in \mathfrak{A}_{m,n}$, $i = 1,\dots,n-1$. The image under the map π, defined in (6.14), of the element $\tilde{u}_{i+1}$ is $\pi(\tilde{u}_{i+1}) = s_i\tilde{j}_i - \tilde{j}s_i \in \mathbb{C}G(m,1,n)$, $i = 1,\dots,n-1$. The action of $\pi(\tilde{u}_{i+1})$ in a representation $V_{\lambda^{(m)}}$ is:

$$\mathcal{X}_{\lambda^{(m)}} \mapsto \Bigl(cc^{(i)} - cc^{(i+1)} - \delta_{p^{(i)},p(i+1)}\Bigr)\,\mathcal{X}^{s_i}_{\lambda^{(m)}}\ , \tag{6.40}$$

where $cc^{(i)} = cc(X_{\lambda^{(m)}}|i)$, $p^{(i)} = p(X_{\lambda^{(m)}}|i)$, $i = 1,\dots,n$; $\delta_{p,p'}$ is the Kronecker symbol. Indeed we rewrite $\tilde{u}_{i+1} = s_i(\tilde{j}_i - \tilde{j}_{i+1}) + \frac{1}{m}\sum\limits_{l=1}^{m} j_i^l j_{i+1}^{-l}$ and so, by the Lemma 28,

$$\tilde{u}_{i+1}(\mathcal{X}_{\lambda^{(m)}}) = (cc^{(i)} - cc^{(i+1)})\left(s_i(\mathcal{X}_{\lambda^{(m)}}) + \frac{\delta_{p^{(i)},p^{(i+1)}}}{cc^{(i)} - cc^{(i+1)}}\mathcal{X}_{\lambda^{(m)}}\right)\ .$$

Using (6.37)-(6.38) we obtain the formula (6.40).

6.8.4 Scalar product

1. The representations of $G(m,1,n)$ given by formulas (6.37)–(6.39) are analogues of the semi-normal representations of the symmetric group. In this Subsection we provide analogues for $G(m,1,n)$ of the orthogonal representations of the symmetric group. The formulas from this Subsection could be obtained by taking the classical limit of the formulas in Subsection 4.4. However the classical limit presents certain subtleties (and the classical formulas become shorter); we give here an independent of the non-degenerate case presentation.

Let $\lambda^{(m)}$ be an m-partition and let $X_{\lambda^{(m)}}$ and $X'_{\lambda^{(m)}}$ be two different standard m-tableaux of shape $\lambda^{(m)}$. For brevity we set $cc^{(i)} = cc(X_{\lambda^{(m)}}|i)$ and $p^{(i)} = p(X_{\lambda^{(m)}}|i)$ for all $i = 1,\dots,n$. Define the following Hermitian scalar product on the vector space $U_{\lambda^{(m)}}$:

$$\langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}'_{\lambda^{(m)}}\rangle = 0\ , \tag{6.41}$$

$$\langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle = \prod_{j,k:\, j<k\,,\, p^{(j)}=p^{(k)}\,,\, cc^{(j)}\notin\{cc^{(k)},cc^{(k)}\pm 1\}} \frac{cc^{(j)}-cc^{(k)}-1}{cc^{(j)}-cc^{(k)}} \ . \tag{6.42}$$

The so defined scalar product is positive definite.

Notice that, if $X^{s_i}_{\lambda^{(m)}}$ is a standard m-tableau, we have

$$\langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}} \rangle = \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle \quad \text{if } p^{(i)} \neq p^{(i+1)}, \tag{6.43}$$

and

$$\langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}} \rangle = -\frac{cc^{(i+1)}-cc^{(i)}-1}{cc^{(i)}-cc^{(i+1)}-1} \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle \quad \text{if } p^{(i)} = p^{(i+1)} \ . \tag{6.44}$$

We shall show that this Hermitian scalar product is invariant under the action of the group $G(m,1,n)$ given by formulas (6.37)–(6.39). It is immediate that (6.41)–(6.42) are invariant under the action of the generator t of $G(m,1,n)$. The verification of the invariance of (6.41) under the action of the generator s_i of $G(m,1,n)$ is non-trivial only if $X^{s_i}_{\lambda^{(m)}}$ is standard and $X'_{\lambda^{(m)}} = X^{s_i}_{\lambda^{(m)}}$. In this situation assume first that $p^{(i)} \neq p^{(i+1)}$. Then

$$\langle s_i(\mathcal{X}_{\lambda^{(m)}}), s_i(\mathcal{X}^{s_i}_{\lambda^{(m)}}) \rangle = \langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle = 0.$$

Now assume that $p^{(i)} = p^{(i+1)}$. It is straightforward to obtain that $\langle s_i(\mathcal{X}_{\lambda^{(m)}}), s_i(\mathcal{X}^{s_i}_{\lambda^{(m)}}) \rangle$ is equal to

$$\frac{(cc^{(i+1)}-cc^{(i)}-1)\langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle + (cc^{(i)}-cc^{(i+1)}-1)\langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}} \rangle}{(cc^{(i)}-cc^{(i+1)})^2}.$$

Using the formula (6.44) we obtain

$$\langle s_i(\mathcal{X}_{\lambda^{(m)}}), s_i(\mathcal{X}^{s_i}_{\lambda^{(m)}}) \rangle = 0 \ ,$$

and the verification that (6.41) is invariant under the action of $G(m,1,n)$ is finished.

If $X^{s_i}_{\lambda^{(m)}}$ is not standard, then $s_i(\mathcal{X}_{\lambda^{(m)}}) = \pm \mathcal{X}_{\lambda^{(m)}}$ and the invariance of (6.42) follows. Assume that $X^{s_i}_{\lambda^{(m)}}$ is standard and that $p^{(i)} \neq p^{(i+1)}$, then $s_i(\mathcal{X}_{\lambda^{(m)}}) = \mathcal{X}^{s_i}_{\lambda^{(m)}}$ and therefore, using (6.43),

$$\langle s_i(\mathcal{X}_{\lambda^{(m)}}), s_i(\mathcal{X}_{\lambda^{(m)}}) \rangle = \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle \ .$$

Now assume that $X^{s_i}_{\lambda^{(m)}}$ is standard and that $p^{(i)} = p^{(i+1)}$. A straightforward calculation leads to

$$\langle s_i(\mathcal{X}_{\lambda^{(m)}}), s_i(\mathcal{X}_{\lambda^{(m)}}) \rangle = \frac{\langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle + (cc^{(i)}-cc^{(i+1)}-1)^2 \langle \mathcal{X}^{s_i}_{\lambda^{(m)}}, \mathcal{X}^{s_i}_{\lambda^{(m)}} \rangle}{(cc^{(i)}-cc^{(i+1)})^2} \ .$$

Using (6.44) one obtains

$$\langle s_i(\mathcal{X}_{\lambda^{(m)}}), s_i(\mathcal{X}_{\lambda^{(m)}}) \rangle = \langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle \ ,$$

which concludes the proof that (6.41)–(6.42) are invariant under the action of the group $G(m,1,n)$.

2. As a consequence, the operators for the elements of $G(m,1,n)$ are unitary in the basis $\{\tilde{\mathcal{X}}_{\lambda^{(m)}}\}$ where

$$\tilde{\mathcal{X}}_{\lambda^{(m)}} := \left(\prod_{j,k:\, j<k\,,\, p^{(j)}=p^{(k)}\,,\, cc^{(j)}\notin\{cc^{(k)},cc^{(k)}\pm 1\}} \left(\frac{cc^{(j)}-cc^{(k)}-1}{cc^{(j)}-cc^{(k)}} \right)^{\frac{1}{2}} \right) \mathcal{X}_{\lambda^{(m)}}$$

for any standard m-tableau $X_{\lambda^{(m)}}$ of shape $\lambda^{(m)}$.

3. Another possible formula for the Hermitian scalar product is, instead of (6.42),

$$\langle \mathcal{X}_{\lambda^{(m)}}, \mathcal{X}_{\lambda^{(m)}} \rangle = \prod_{j,k:\, j<k\,,\, p^{(j)}=p^{(k)}\,,\, cc^{(j)}\notin\{cc^{(k)},cc^{(k)}\pm 1\}} |cc^{(j)}-cc^{(k)}-1| \ . \tag{6.45}$$

The right hand sides in the two formulas (6.42) and (6.45) differ only by a factor equal to

$$\prod_{j,k:\, j<k\,,\, p^{(j)}=p^{(k)}\,,\, cc^{(j)}\notin\{cc^{(k)},cc^{(k)}\pm 1\}} |cc^{(j)}-cc^{(k)}| \ .$$

This factor does not depend on the particular m-tableau $X_{\lambda^{(m)}}$ of shape $\lambda^{(m)}$; it only depends on the m-partition $\lambda^{(m)}$.

4. The formula (6.45) can be rewritten without absolute values in the following way. Define

$$\Upsilon(X_{\lambda^{(m)}}) := \prod_{j,k:\, j<k\,,\, p^{(j)}=p^{(k)}\,,\, cc^{(j)}\notin\{cc^{(k)},cc^{(k)}\pm 1\}} (cc(X_{\lambda^{(m)}}|j) - cc(X_{\lambda^{(m)}}|k)) \ .$$

Then

$$\Upsilon(X'_{\lambda^{(m)}}) = (-1)^{\ell(w)}\ \Upsilon(X_{\lambda^{(m)}}) \ ,$$

where $X_{\lambda^{(m)}}$ and $X'_{\lambda^{(m)}}$ are two standard m-tableaux of the same shape $\lambda^{(m)}$ and $\ell(w)$ is the length of the permutation w which transforms $X_{\lambda^{(m)}}$ into $X'_{\lambda^{(m)}}$. Thus if we fix one standard tableau $X^{\circ}_{\lambda^{(m)}}$ of the shape $\lambda^{(m)}$ then

$$(-1)^{\ell(w(X_{\lambda^{(m)}}))}\ \Upsilon(X_{\lambda^{(m)}}) \ ,$$

where $X_{\lambda^{(m)}}$ is another standard tableau of the shape $\lambda^{(m)}$ and $w(X_{\lambda^{(m)}})$ is the permutation transforming $X^{\circ}_{\lambda^{(m)}}$ into $X_{\lambda^{(m)}}$, has the same sign for all $X_{\lambda^{(m)}}$. Let ε be the sign of $\Upsilon(X^{\circ}_{\lambda^{(m)}})$. Then the right hand side of (6.45) equals

$$\varepsilon\,(-1)^{\ell(w(X_{\lambda^{(m)}}))}\ \Upsilon(X_{\lambda^{(m)}}) \ .$$

6.9 Completeness

Here are the analogues of the results of Section 5 for the classical limit. The proofs are completely similar to the proofs in Section 5.

Proposition 29. *The set* Spec $\begin{pmatrix} j_1 & , \dots , & j_n \\ \tilde{j}_1 & , \dots , & \tilde{j}_n \end{pmatrix}$, *the set* $\mathrm{cCont}_m(n)$ *and the set of standard m-tableaux are in bijection.*

Corollary 30. *The spectrum of the classical Jucys–Murphy elements is simple in the representations $V_{\lambda^{(m)}}$ (labeled by the m-partitions).*

It means that for two different standard m-tableaux (not necessarily of the same shape) the elements of Spec $\begin{pmatrix} j_1 & ,\dots, & j_n \\ \tilde{j}_1 & ,\dots, & \tilde{j}_n \end{pmatrix}$ associated to them by the Proposition 29 are different (two arrays $\begin{pmatrix} a_1 & ,\dots, & a_n \\ \tilde{a}_1 & ,\dots, & \tilde{a}_n \end{pmatrix}$ and $\begin{pmatrix} a'_1 & ,\dots, & a'_n \\ \tilde{a}'_1 & ,\dots, & \tilde{a}'_n \end{pmatrix}$ are different if there is some i such that $a_i \neq a'_i$ or $\tilde{a}_i \neq \tilde{a}'_i$).

It remains to verify that we obtain within this approach all irreducible representations of the group $G(m,1,n)$. According to Appendix B the sum of the squares of the dimensions of the constructed representations equals the order of $G(m,1,n)$. Thus the following Proposition completes the verification.

Proposition 31. *The representations $V_{\lambda^{(m)}}$ (labeled by the m-partitions) of the group $G(m,1,n)$ constructed in the preceding subsection are irreducible and pairwise non-isomorphic.*

As a result, the branching rules for the pair $(G(m,1,n), G(m,1,n-1))$ (that is, for the pair $(\mathbb{C}G(m,1,n), \mathbb{C}G(m,1,n-1))$ of algebras) are the same as in the non-degenerate situation, for the pair $(H(m,1,n), H(m,1,n-1))$.

Similarly to the non-degenerate situation, we obtain the following conclusions.

- The branching rules for the chain, with respect to n, of the groups $G(m,1,n)$ are free of multiplicities.
- The centralizer of the sub-algebra $\mathbb{C}G(m,1,n-1)$ in the algebra $\mathbb{C}G(m,1,n)$ is commutative for each $n = 1,2,3,\dots$
- The centralizer of the subalgebra $\mathbb{C}G(m,1,n-1)$ in the algebra $\mathbb{C}G(m,1,n)$ is generated by the center of $\mathbb{C}G(m,1,n-1)$ and the Jucys–Murphy elements j_n and $\tilde{j}_n$.
- The subalgebra generated by the Jucys–Murphy elements $j_1,\dots,j_n,\tilde{j}_1,\dots,\tilde{j}_n$ of the algebra $\mathbb{C}G(m,1,n)$ is maximal commutative.

Remark. This remark repeats, in the classical situation, the remark **(a)** at the end of Section 5. For every standard m-tableau $X_{\lambda^{(m)}}$ define the element $\mathfrak{p}_{X_{\lambda^{(m)}}}$ of the ring $\mathbb{C}G(m,1,n)$ by the following recursion. The initial condition is $\mathfrak{p}_\varnothing = 1$. Let $\alpha^{(m)}$ be the m-node occupied by the number n in $X_{\lambda^{(m)}}$;

define $\mu^{(m)} := \lambda^{(m)} \backslash \{\alpha^{(m)}\}$. Let $X_{\mu^{(m)}}$ be the standard m-tableau with the numbers $1, \dots, n-1$ at the same m-nodes as in $X_{\lambda^{(m)}}$. Then the recursion is given by

$$\mathfrak{p}_{X_{\lambda^{(m)}}} := \mathfrak{p}_{X_{\mu^{(m)}}} \prod_{\substack{\beta^{(m)}: \\ \beta^{(m)} \in \mathcal{E}_+(\mu^{(m)}) \\ cc(\beta^{(m)}) \neq cc(\alpha^{(m)})}} \frac{\tilde{j}_n - cc(\beta^{(m)})}{cc(\alpha^{(m)}) - cc(\beta^{(m)})} \prod_{\substack{\beta^{(m)}: \\ \beta^{(m)} \in \mathcal{E}_+(\mu^{(m)}) \\ p(\beta^{(m)}) \neq p(\alpha^{(m)})}} \frac{j_n - p(\beta^{(m)})}{p(\alpha^{(m)}) - p(\beta^{(m)})} , \quad (6.46)$$

where $\begin{pmatrix} p(\beta^{(m)}) \\ cc(\beta^{(m)}) \end{pmatrix}$ is the classical content of the m-node $\beta^{(m)}$. Due to the completeness results of this Section, the elements $\mathfrak{p}_{X_{\lambda^{(m)}}}$ form a complete set of pairwise orthogonal primitive idempotents of the algebra $\mathbb{C}G(m,1,n)$.

As for the cyclotomic Hecke algebra, we have a well-defined homomorphism $\mathfrak{T}_c \to \mathbb{C}G(m,1,n)$ which is identical on the generators $t, s_1, \dots, s_{n-1}$ and sends $\mathcal{X}_{\lambda^{(m)}}$ to $\mathfrak{p}_{X_{\lambda^{(m)}}}$ for all standard m-tableaux $X_{\lambda^{(m)}}$. The verification is similar to the one for the algebra $H(m,1,n)$ (see the remark **(a)** at the end of Section 5). We leave it to the reader.

Appendix 6.A Structure of degenerate cyclotomic affine Hecke algebra

Here we describe a normal form for elements of the degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$, see the Definition 17. It coincides with the one given in [39] for the wreath Hecke algebra if we take, for the finite group in [39], the cyclic group of order m. Nevertheless we sketch here for completeness the proof in our particular situation. Fix any basis $\mathfrak{B}$ in the group ring of the group $G(m,1,n)$. Recall the injective homomorphism $\hat{\iota}: \mathbb{C}G(m,1,n) \to \mathfrak{A}_{m,n}$ defined in (6.13).

Proposition 32. *The following set is a basis of* $\mathfrak{A}_{m,n}$*:*

$$\tilde{x}_1^{k_1} \dots \tilde{x}_n^{k_n} \cdot \overline{w} , \quad (6.47)$$

where $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ *and* $\overline{w} \in \hat{\iota}(\mathfrak{B})$*; here* $\mathbb{Z}_{\geq 0}$ *is the set of non-negative integers.*

Sketch of the proof. The defining relations (6.10)–(6.12), together with the definition (6.16), the Proposition 19 and the Lemma 20 imply that any element of $\mathfrak{A}_{m,n}$ can be written as a linear combination of the elements (6.47). Only the linear independence of the elements (6.47) needs some care. Let E be the vector space with the basis

$$\tilde{u}_1^{k_1} \dots \tilde{u}_n^{k_n} , \quad \text{where } k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}. \quad (6.48)$$

The element corresponding to $k_1 = \dots = k_n = 0$ we denote by 1.

Define the operators $L_{\tilde{x}_i}$, $i = 1, \dots, n$, on E:

$$L_{\tilde{x}_i}(\tilde{u}_1^{k_1} \dots \tilde{u}_i^{k_i} \dots \tilde{u}_n^{k_n}) = \tilde{u}_1^{k_1} \dots \tilde{u}_i^{k_i+1} \dots \tilde{u}_n^{k_n} . \quad (6.49)$$

Clearly $L_{\tilde{x}_i}$, $i = 1, \dots, n$, form a commutative set of operators.

Let V be a left regular $G(m,1,n)$-module. The basis $\mathfrak{B}$ induces a basis in V which we denote by the same symbol $\mathfrak{B}$. Let $F := E \otimes V$. Extend the operators $L_{\tilde{x}_i}$, $i = 1, \dots, n$, to the operators on F acting as the identity on V. We have

$$L_{\tilde{x}_1}^{k_1} \dots L_{\tilde{x}_n}^{k_n}(1 \otimes v) = \tilde{u}_1^{k_1} \dots \tilde{u}_n^{k_n} \otimes v \ , \tag{6.50}$$

where v is an arbitrary element of V.

Define the operators $L_{\overline{s}_i}$, $i = 1, \dots, n-1$ and L_{x_i}, $i = 1, \dots, n$ on F by the rules:

- The action of $L_{\overline{s}_i}$, $i = 1, \dots, n-1$, (respectively, L_{x_i}, $i = 1, \dots, n$) on the subspace formed by the elements $1 \otimes v$, $v \in V$, is the left regular action of the generator s_i (respectively, of the element j_i) of the group $G(m,1,n)$.
- To define the result of the action of $L_{\overline{s}_i}$, $i = 1, \dots, n-1$ and L_{x_i}, $i = 1, \dots, n$ on an element $\tilde{u}_1^{k_1} \dots \tilde{u}_n^{k_n} \otimes v$, we use (6.50) to write

$$\tilde{u}_1^{k_1} \dots \tilde{u}_n^{k_n} \otimes v = L_{\tilde{x}_1}^{k_1} \dots L_{\tilde{x}_n}^{k_n}(1 \otimes v) \tag{6.51}$$

  and then move $L_{\overline{s}_i}$ (respectively, L_{x_i}) to the right through $L_{\tilde{x}_1}^{k_1} \dots L_{\tilde{x}_n}^{k_n}$ postulating the following commutation relations:

$$\begin{gathered} L_{\overline{s}_i} L_{\tilde{x}_i} = L_{\tilde{x}_{i+1}} L_{\overline{s}_i} - \tfrac{1}{m} \textstyle\sum_{p=1}^m L_{x_i}^p L_{x_{i+1}}^{-p} \ , \ L_{\overline{s}_i} L_{\tilde{x}_{i+1}} = L_{\tilde{x}_i} L_{\overline{s}_i} + \tfrac{1}{m} \textstyle\sum_{p=1}^m L_{x_i}^p L_{x_{i+1}}^{-p} \ , \\ L_{\overline{s}_i} L_{\tilde{x}_j} = L_{\tilde{x}_j} L_{\overline{s}_i} \ \text{ if } \ j \neq i, i+1 \ , \\ L_{x_i} L_{\tilde{x}_j} = L_{\tilde{x}_j} L_{x_i} \ \text{ for all } \ i,j = 1, \dots, n \ . \end{gathered} \tag{6.52}$$

In the process of moving $L_{\overline{s}_i}$ to the right through $L_{\tilde{x}_1}^{k_1} \dots L_{\tilde{x}_n}^{k_n}$, elements L_{x_i} and $L_{x_{i+1}}$ appear. We move them to the right as well using the same rules (6.52).

The resulting explicit formula for the action of the operators $L_{\overline{s}_i}$, L_{x_i} on the space F is:

$$\begin{aligned} L_{\overline{s}_i}(\tilde{u}_1^{k_1} \dots \tilde{u}_i^{k_i} \tilde{u}_{i+1}^{k_{i+1}} \dots \tilde{u}_n^{k_n} \otimes v) = \ & \tilde{u}_1^{k_1} \dots \tilde{u}_i^{k_{i+1}} \tilde{u}_{i+1}^{k_i} \dots \tilde{u}_n^{k_n} \otimes s_i v \\ + \sum_{a=1}^{k_{i+1}} \tilde{u}_1^{k_1} \dots \tilde{u}_i^{a-1} \tilde{u}_{i+1}^{k_i+k_{i+1}-a} \dots \tilde{u}_n^{k_n} \otimes \Pi_i v & - \sum_{a=1}^{k_i} \tilde{u}_1^{k_1} \dots \tilde{u}_i^{a-1} \tilde{u}_{i+1}^{k_i+k_{i+1}-a} \dots \tilde{u}_n^{k_n} \otimes \Pi_i v \ , \end{aligned}$$

where $\Pi_i = \frac{1}{m} \sum\limits_{p=1}^{m} j_i^p j_{i+1}^{-p}$, and

$$L_{x_i}(\tilde{u}_1^{k_1} \dots \tilde{u}_n^{k_n} \otimes v) \ = \ \tilde{u}_1^{k_1} \dots \tilde{u}_n^{k_n} \otimes j_i v \ .$$

One checks that the operators L_{x_i}, $L_{\tilde{x}_i}$, $i = 1, \dots, n$, and $L_{\overline{s}_i}$, $i = 1, \dots, n-1$, verify the defining relations (6.10)–(6.12) on the whole space F and define thereby the $\mathfrak{A}_{m,n}$-module structure on the space F.

Therefore,

$$L_{\tilde{x}_1^{k_1}\ldots\tilde{x}_n^{k_n}\,\cdot\,\overline{w}}(1\otimes 1) \;=\; L_{\tilde{x}_1}^{k_1}\ldots L_{\tilde{x}_n}^{k_n}\;\cdot\;L_{\overline{w}}(1\otimes 1) \;=\; \tilde{u}_1^{k_1}\ldots\tilde{u}_n^{k_n}\otimes w\ ,$$

where $w\in\mathfrak{B}$ and $\overline{w}$ is the image of w under the map $\hat{\imath}$; this shows that the operators $L_{\tilde{x}_1^{k_1}\ldots\tilde{x}_n^{k_n}\cdot\overline{w}}$ are linearly independent and the linear independence of the set (6.47) follows. □

Appendix 6.B Classical intertwining operators

Here we describe the intertwining operators in the degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$; they can be used to investigate the spectrum of the elements $\tilde{x}_i$ in different representations. We discuss the origin of these intertwining operators in the (non-degenerate) affine Hecke algebra. We also rederive the spectrum of the Jucys–Murphy elements $\tilde{j}_i$ from the perturbation theory point of view. The intertwining operators can be introduced [39] in the more general context of the wreath Hecke algebra.

1. The following Proposition is the classical analogue of the Proposition 4.

Proposition 33. *We have*

$$\mathrm{Spec}(\tilde{j}_i)\subset[1-i,i-1]\quad \textit{for all}\ \ i=1,\ldots,n. \tag{6.53}$$

The Proposition 33 follows from the Propositions 21 and 23 like the Proposition 4 follows from the Propositions 1 and 3 in the proof given in paragraph 5 in Section 3.

To give an alternative proof (in the spirit of [14]), mentioned in paragraph 5 in Section 3, we introduce the following elements of the algebra $\mathfrak{A}_{m,n}$:

$$\tilde{u}_{i+1}:=\overline{s}_i\tilde{x}_i-\tilde{x}_i\overline{s}_i\equiv\overline{s}_i(\tilde{x}_i-\tilde{x}_{i+1})+P_{i+1}\ ,\quad i=1,\ldots,n-1, \tag{6.54}$$

where we denoted $P_{i+1}:=\frac{1}{m}\sum_{p=1}^{m}\ x_i^p x_{i+1}^{-p}$. Clearly, the elements P_{i+1} are idempotents

$$P_{i+1}^2=P_{i+1} \tag{6.55}$$

and satisfy

$$\left\{\begin{array}{ll}(x_i-x_{i+1})P_{i+1}=0\ , & (6.56)\\[2mm] \overline{s}_iP_{i+1}=P_{i+1}\overline{s}_i\ . & (6.57)\end{array}\right.$$

The elements $\tilde{u}_i$ are the "classical intertwining" operators, they satisfy (the verification is straightforward)

$$\left\{\begin{array}{l}\tilde{u}_{i+1}x_i=x_{i+1}\tilde{u}_{i+1}\ ,\quad \tilde{u}_{i+1}x_{i+1}=x_i\tilde{u}_{i+1}\ ,\ \tilde{u}_{i+1}x_j=x_j\tilde{u}_{i+1}\ \text{ for }\ j\neq i,i+1\ ,\\[2mm] \tilde{u}_{i+1}\tilde{x}_i=\tilde{x}_{i+1}\tilde{u}_{i+1}\ ,\quad \tilde{u}_{i+1}\tilde{x}_{i+1}=\tilde{x}_i\tilde{u}_{i+1}\ ,\ \tilde{u}_{i+1}\tilde{x}_j=\tilde{x}_j\tilde{u}_{i+1}\ \text{ for }\ j\neq i,i+1\ .\end{array}\right. \tag{6.58}$$

Next, the elements $\tilde{u}_i$ satisfy the Artin relations:

$$\tilde{u}_i \tilde{u}_{i+1} \tilde{u}_i = \tilde{u}_{i+1} \tilde{u}_i \tilde{u}_{i+1} \ . \tag{6.59}$$

In the verification here it is convenient to use (6.58) to transform, say, the left hand side of (6.59), starting as follows:

$$\tilde{u}_i \tilde{u}_{i+1} \tilde{u}_i = \Big(\overline{s}_{i-1}(\tilde{x}_{i-1} - \tilde{x}_i) + P_i\Big) \tilde{u}_{i+1} \tilde{u}_i = \overline{s}_{i-1} \tilde{u}_{i+1} \tilde{u}_i (\tilde{x}_i - \tilde{x}_{i+1}) + P_i \tilde{u}_{i+1} \tilde{u}_i \ .$$

Continuing this way, we move out to the right all $\tilde{x}$'s; the comparison of the left hand side and the right hand side is more or less direct afterwards. In the comparison, the equalities $\overline{s}_i P_i \overline{s}_i = \overline{s}_{i-1} P_{i+1} \overline{s}_{i-1}$, $(\frac{1}{m}\sum_{p=1}^{m} \ x_{i-1}^p x_{i+1}^{-p} - P_{i+1})P_i = 0$ and $(\frac{1}{m}\sum_{p=1}^{m} \ x_{i-1}^p x_{i+1}^{-p} - P_i)P_{i+1} = 0$ are useful; the first equality is a straightforward calculation, the second and third equalities follow from (6.56).

One more property of the elements $\tilde{u}_i$ is

$$\tilde{u}_{i+1}^2 = -(\tilde{x}_i - \tilde{x}_{i+1})^2 + P_{i+1} = -\Big(\tilde{x}_i - \tilde{x}_{i+1} + P_{i+1}\Big)\Big(\tilde{x}_i - \tilde{x}_{i+1} - P_{i+1}\Big) \ . \tag{6.60}$$

The relation (6.60) can be verified directly or following the method above for the verification of the relation (6.59).

Therefore, for a polynomial χ in one variable, we have

$$\tilde{u}_{i+1} \chi(\tilde{x}_i) \tilde{u}_{i+1} = \chi(\tilde{x}_{i+1}) \tilde{u}_{i+1}^2 = -\chi(\tilde{x}_{i+1})\Big(\tilde{x}_i - \tilde{x}_{i+1} + P_{i+1}\Big)\Big(\tilde{x}_i - \tilde{x}_{i+1} - P_{i+1}\Big) \ . \tag{6.61}$$

The elements $\tilde{x}_i$, $\tilde{x}_{i+1}$ and P_{i+1} commute. In a representation ρ, the spectrum of the operator $\rho(P_{i+1})$ is contained in $\{0,1\}$; taking for χ the characteristic equation for $\rho(\tilde{x}_i)$, we conclude that

$$\mathrm{Spec}(\rho(\tilde{x}_{i+1})) \subset \mathrm{Spec}(\rho(\tilde{x}_i)) \cup \Big(\mathrm{Spec}(\rho(\tilde{x}_i)) + 1\Big) \cup \Big(\mathrm{Spec}(\rho(\tilde{x}_i)) - 1\Big) \ . \tag{6.62}$$

Realizing $\tilde{x}_i$ by $\tilde{j}_i$ in a representation of the group $G(m,1,n)$ and taking into account the "initial condition" $\tilde{j}_1 = 0$ we rederive (6.53).

Remark. The usual degenerate affine Hecke algebra (it corresponds to $m = 1$) is distinguished in the sense that the idempotents P_i become trivial: $P_i = 1$, in contrast to the degenerate cyclotomic affine Hecke algebras with $m > 1$.

2. Let $\hat{H}_n$ be the affine Hecke algebra. We shall denote the generators of $\hat{H}_n$ by $\sigma_1, \dots, \sigma_{n-1}$ and y_1, the symbol y_1 is used instead of τ here. Denote by y_i the Jucys–Murphy operators of the algebra $\hat{H}_n$; recall that y_{i+1} for $i > 0$ are defined inductively by $y_{i+1} := \sigma_i y_i \sigma_i$.

General intertwining operators $\mathfrak{U}_{i+1}$, $i = 1, \dots, n-1$, of the affine Hecke algebra are defined to be operators which verify

$$\begin{cases} \mathfrak{U}_{i+1} y_i = y_{i+1} \mathfrak{U}_{i+1} \ , \ \mathfrak{U}_{i+1} y_{i+1} = y_i \mathfrak{U}_{i+1} \ , \\ \mathfrak{U}_{i+1} y_k = y_k \mathfrak{U}_{i+1} \quad \text{for} \quad k \neq i, i+1 \end{cases} \tag{6.63}$$

for all $i = 1, \dots, n-1$.

The intertwining operators (the solutions $\mathfrak{U}_{i+1} := U_{i+1}$ of (6.63)) used in [14] are

$$U_{i+1} := \sigma_i y_i - y_i \sigma_i \ . \tag{6.64}$$

The operators U_{i+1} satisfy, in addition to (6.63), to the Yang–Baxter equation,

$$U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1} \ , \tag{6.65}$$

and square to the following function of the Jucys–Murphy elements:

$$U_i^2 = -(y_{i+1} - q^2 y_i)(y_{i+1} - q^{-2} y_i) \ . \tag{6.66}$$

In contrast to (6.60), the right hand side of (6.66) does not contain anything analogous to the projector P_{i+1}. We shall explain the appearance of the projectors in the classical limit.

We could work directly with the Jucys–Murphy operators of the cyclotomic algebra $H(m,1,n)$ and use the formulas (6.8)-(6.9). However, one can work (and we prefer to do so) on the level of the affine Hecke algebra and take the cyclotomic quotient afterwards. Namely, we have the following composition:

$$\hat{H}_n \to H(m,1,n) \rightsquigarrow \mathbb{C}G(m,1,n) \tag{6.67}$$

Here the homomorphism from $\hat{H}_n$ to $H(m,1,n)$ is given by $\hat{H}_n \ni \sigma_i \mapsto \sigma_i \in H(m,1,n)$ and $y_1 \mapsto \tau$, where τ satisfies (2.8). The symbol $\rightsquigarrow$ stands for the classical limit. The classical limit can be also understood as a homomorphism like other arrows in (6.67); namely, the classical limit here is the quotient by the ideal generated by $\{v_i - \xi_i, i = 1, \dots, n\}$ and $(q-1)$, where v_i and q are considered as central generators.

The formulas (6.8)-(6.9) show that the Taylor series decompositions of the Jucys–Murphy operators of the cyclotomic algebra $H(m,1,n)$ begin as follows

$$J_i = j_i + j_i \tilde{j}_i \alpha + O(\alpha^2) \ ; \tag{6.68}$$

here α is the deformation parameter, $q^2 = 1 + \alpha + O(\alpha^2)$. We "lift" the formula (6.68) to the affine Hecke algebra by assuming that the Taylor series decompositions of the Jucys–Murphy operators of the affine algebra $\hat{H}_n$ begin as follows

$$y_i = x_i + x_i \tilde{x}_i \alpha + O(\alpha^2) \ , \tag{6.69}$$

where x_i and $\tilde{x}_i$ belong to the degenerate cyclotomic affine Hecke algebra $\mathfrak{A}_{m,n}$, see (6.10)-(6.12). The assumption (6.69) can be understood as a sort of considering the "first infinitesimal neighborhood" of the homomorphism from $\hat{H}_n$ to $\mathbb{C}G(m,1,n)$ participating in (6.67).

To perform the classical limit one takes into account the following beginning of the Taylor series decomposition of the elements σ_i:

$$\sigma_i = s_i + \frac{\alpha}{2} + O(\alpha^2) \ . \tag{6.70}$$

Note that the operators U_{i+1}, given by (6.64) tend, by (6.69) and (6.70), to the operators

$$u_{i+1} := \overline{s}_i x_i - x_i \overline{s}_i \equiv \overline{s}_i (x_i - x_{i+1}) \ . \tag{6.71}$$

The operators u_{i+1} satisfy all the relations for the operators $\tilde{u}_{i+1}$ listed in (6.58); but the intertwining operators u_{i+1} do not help to understand the spectrum of the images of the elements $\tilde{x}_i$ in some representation.

As noted in [14], the operators $\mathfrak{U}_{i+1} := U_{i+1} f(y_i, y_{i+1})$, where f is an arbitrary function, are intertwining operators which satisfy the Yang–Baxter equation.

One shows by induction that for any positive integer L,

$$\sigma_i y_i^L - y_i^L \sigma_i = U_{i+1} \cdot \sum_{b=0}^{L-1} y_i^{\,b}\, y_{i+1}^{L-1-b} \ . \tag{6.72}$$

Therefore, the operators $\sigma_i y_i^L - y_i^L \sigma_i$ are intertwining operators for any positive integer L.

Under the assumption (6.69), the operators $\sigma_i y_i^m - y_i^m \sigma_i \equiv \sigma_i (y_i^m - 1) - (y_i^m - 1)\sigma_i$ tend to 0 when α tends to 0. These operators are of order $O(\alpha)$. Denote

$$\tilde{\mathcal{U}}_{i+1} := \frac{1}{m}\Bigl(\sigma_i \, \frac{y_i^m - 1}{\alpha} - \frac{y_i^m - 1}{\alpha}\, \sigma_i\Bigr) \ . \tag{6.73}$$

Clearly, $\tilde{\mathcal{U}}_{i+1}$ tend to $\tilde{u}_{i+1}$ when α tends to 0 ($q - q^{-1} = \alpha + O(\alpha^2)$).

Using (6.72) with $L = m$, (6.63) with $\mathfrak{U}_{i+1} := U_{i+1}$ and (6.66), we obtain

$$\begin{array}{rcl}
(\sigma_i y_i^m - y_i^m \sigma_i)^2 & = & U_{i+1} \cdot \displaystyle\sum_{b=0}^{m-1} y_i^b\, y_{i+1}^{m-1-b} \cdot U_{i+1} \cdot \sum_{c=0}^{m-1} y_i^c\, y_{i+1}^{m-1-c} \\
 & = & U_{i+1}^2 \cdot \Bigl(\displaystyle\sum_{b=0}^{m-1} y_i^b\, y_{i+1}^{m-1-b}\Bigr)^2 \\
 & = & -(y_{i+1} - q^2 y_i)(y_{i+1} - q^{-2} y_i)\Bigl(\displaystyle\sum_{b=0}^{m-1} y_i^b\, y_{i+1}^{m-1-b}\Bigr)^2 \\
 & = & -\Bigl((y_{i+1} - q^2 y_i)\displaystyle\sum_{b=0}^{m-1} y_i^b\, y_{i+1}^{m-1-b}\Bigr) \cdot \Bigl((y_{i+1} - q^{-2} y_i)\sum_{b=0}^{m-1} y_i^b\, y_{i+1}^{m-1-b}\Bigr) \ .
\end{array} \tag{6.74}$$

Let $r := x_i / x_{i+1}$. Under the assumption (6.69) we calculate

$$\sum_{b=0}^{m-1} y_i^b\, y_{i+1}^{m-1-b} = m x_i^{m-1} P_{i+1} + \alpha\Bigl(\tilde{x}_i x_{i+1}^{m-1} \sum_{b=0}^{m-1} b r^b + \tilde{x}_{i+1} x_i^{m-1} \sum_{b=0}^{m-1} b r^{-b}\Bigr) + O(\alpha^2) \tag{6.75}$$

and

$$\left\{ \begin{array}{rcl}
y_{i+1} - q^2 y_i & = & x_{i+1} - x_i + \alpha\Bigl((x_{i+1}\tilde{x}_{i+1} - x_i \tilde{x}_i) - x_i\Bigr) + O(\alpha^2) \ , \\
y_{i+1} - q^{-2} y_i & = & x_{i+1} - x_i + \alpha\Bigl((x_{i+1}\tilde{x}_{i+1} - x_i \tilde{x}_i) + x_i\Bigr) + O(\alpha^2) \ .
\end{array} \right. \tag{6.76}$$

Writing $x_{i+1}-x_i=x_{i+1}(1-r)$ or $x_{i+1}-x_i=-x_i(1-r^{-1})$ and using the identity

$$(1-\mathfrak{c})\sum_{b=0}^{m-1} b\,\mathfrak{c}^b = m(P-1) \tag{6.77}$$

(where $P:=\frac{1}{m}\sum_{b=0}^{m-1}\mathfrak{c}^b$ is the projector), valid for a generator $\mathfrak{c}$ of the cyclic group C_m with m elements, we find

$$\left\{\begin{array}{lcl} (y_{i+1}-q^2y_i)\displaystyle\sum_{b=0}^{m-1} y_i^b\,y_{i+1}^{m-1-b} & = & m\,\alpha(\tilde{x}_{i+1}-\tilde{x}_i-P_{i+1})+O(\alpha^2)\ , \\ (y_{i+1}-q^{-2}y_i)\displaystyle\sum_{b=0}^{m-1} y_i^b\,y_{i+1}^{m-1-b} & = & m\,\alpha(\tilde{x}_{i+1}-\tilde{x}_i+P_{i+1})+O(\alpha^2)\ . \end{array}\right. \tag{6.78}$$

Substituting (6.78) into (6.74), dividing by $(m\,\alpha)^2$ and taking the limit $\alpha\to 0$, we recover the result (6.60) from the perturbative point of view.

3. The elements j_i verify $j_i^m=1$, the characteristic equations for the elements j_i are not significant on the classical level. It is easy to obtain the characteristic equation for $\tilde{j}_i$ starting from a characteristic equation for J_i. Let A_0 be a semi-simple operator on a vector space V. Consider a perturbation of A_0 of the form

$$A=A_0+A_0A_1\alpha+O(\alpha^2)\ , \tag{6.79}$$

where A_1 is also semi-simple and the operators A_0 and A_1 commute. Let $\mathfrak{r}$ be an eigenvalue of A_0 and $V_{\mathfrak{r}}$ the corresponding eigenspace. The operator $A(\alpha)$ on the space $V_{\mathfrak{r}}$ has, up to the order α^2, the form $\mathfrak{r}\,\mathrm{Id}+\mathfrak{r}A_1$, and its eigenvalues are $\mathfrak{r}+\mathfrak{r}\mathfrak{s}_l\alpha$ where $\{\mathfrak{s}_l\}$ is the set of eigenvalues of the restriction of A_1 to $V_{\mathfrak{r}}$.

In the particular situation with $A_0=j_i$, $A_1=\tilde{j}_i$ and $A=J_i$, the spectrum of A is, in general, a subset of $\{v_lq^{2\eta},\ l=1,\dots,m \text{ and } \eta\in\{1-i,\dots,i-1\}\}$. We first take the limit $v_l\to\xi_l$, $l=1,\dots,m$. Then $\xi_lq^{2\eta}=\xi_l+\xi_l\eta\alpha+O(\alpha^2)$ (since $q^2=1+\alpha+O(\alpha^2)$) thus the spectrum of the operator $\tilde{j}_i$ is a subset of $[1-i,i-1]$ and we recover the Proposition 33 from a perturbative point of view.

Appendix A. Normal form for $H(m,1,n)$

We present the Coxeter-Todd algorithm for the chain (with respect to n) of the groups $G(m,1,n)$. We establish the resulting normal form (which differs from the normal form in [1]) for elements of $G(m,1,n)$. This normal form suggests a basis for the algebra $H(m,1,n)$. We show that this is indeed a basis. Several known facts about the chain (with respect to n) of the algebras $H(m,1,n)$ are reestablished with the help of this basis. In particular, we show that $H(m,1,n)$ is a flat deformation of the group ring of $G(m,1,n)$. This is proved in [1] with the use of the representation theory. Our proof does not refer to the representation theory; it is done more in a spirit of classical proofs for the Hecke algebra.

A.1 Coxeter–Todd algorithm for the chain of groups $G(m,1,n)$

For a finite group G given by generators and relations and a subgroup W of G generated by some subset of the generators of G, the Coxeter-Todd algorithm consists [7] in constructing the set of the left cosets of W in G and the action of the generators on this set. To every coset, a vertex in the Coxeter-Todd figure is associated; arrows stand for the action of the generators. The algorithm starts with the left coset $eW = W$ (e is the identity element); only the generators of G which are not in W act non-trivially on this coset producing new vertices. At each step, using the relations of a given presentation we analyze the action of the generators on vertices which are already in the figure and add or erase possible cosets. The algorithm terminates when we know the action of all generators on every coset in the figure. The figure gives an upper bound for the order of a group G. The Coxeter-Todd algorithm lists the set of cosets and provides thereby a "normal form of an element of G with respect to W".

1. Coxeter–Todd figure for the chain. Recall that the complex reflection group $G(m,1,n)$ is generated by the elements $t, s_1, \dots, s_{n-1}$ with the relations (6.2)–(6.3). Let W be its subgroup generated by the elements $t, s_1, \dots, s_{n-2}$. Here we present the Coxeter-Todd algorithm for the group $G(m,1,n)$ with respect to its subgroup W.

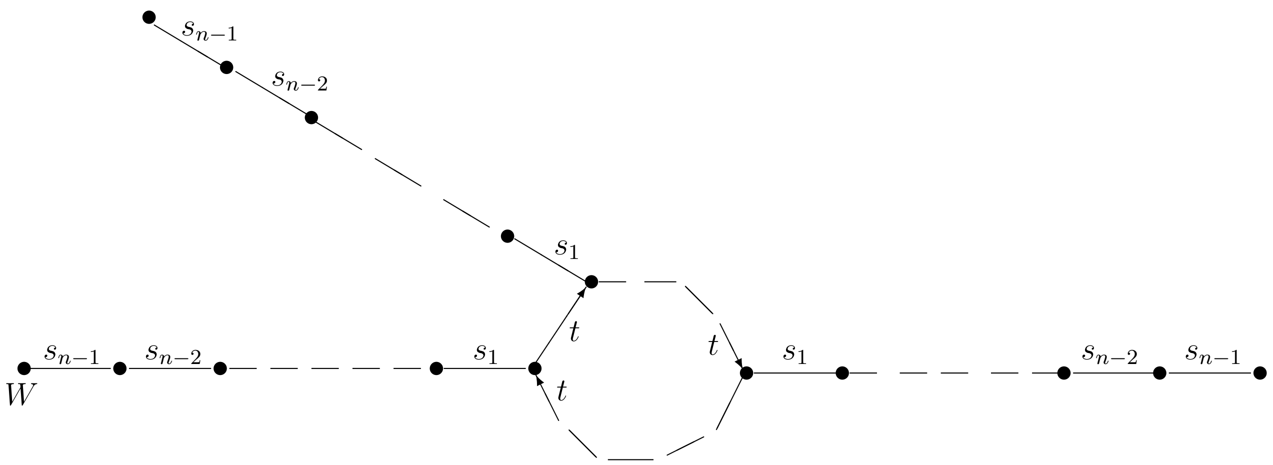

Fig. 1. *Coxeter-Todd figure for $G(m,1,n)$ with respect to W*

The action is indicated by oriented edges (labeled by the generators). For a generator of order 2 (these are the generators $s_1, \dots, s_{n-1}$ of $G(m,1,n)$ if $m \neq 2$) an unoriented edge represents a pair of edges with opposite orientations. If a generator leaves a coset invariant, the corresponding edge is a loop which starts and ends at the vertex representing the coset. For brevity, these loops are omitted; only non-trivial actions are drawn.

In the middle of Fig. 1 there is an m-gon with edges labeled by t. At each vertex of the m-gon starts a tail with $n-1$ edges (labeled in the same way for the tails from all vertices of the m-gon).

To illustrate the utility of the figure, we recalculate the order of the group $G(m,1,n)$, defined by generators and relations. The Fig. 1 has mn vertices; the group $G(m,1,n)$ acts on the set of vertices and the stabilizator of the vertex named W is the subgroup W itself; so the order of $G(m,1,n)$ is $mn|W|$. Clearly, W is a quotient of $G(m,1,n-1)$. By the induction hypothesis[4], $|G(m,1,n-1)| = (n-1)!m^{n-1}$, thus $|W| \leq (n-1)!m^{n-1}$ and $|G(m,1,n)| \leq n!m^n$. On the other hand the standard surjection (recalled in Section 6, see (6.4)) from $G(m,1,n)$ to $C_m \wr S_n$ leads to the opposite inequality $|G(m,1,n)| \geq n!m^n$ and we conclude that

(a) the order $|G(m,1,n)|$ of $G(m,1,n)$ is $n!m^n$,

(b) the surjection from $G(m,1,n)$ to $C_m \wr S_n$ is an isomorphism

and

(c) W is isomorphic to $G(m,1,n-1)$.

2. Normal form for elements of $G(m,1,n)$. Due to the algorithm we have a normal form for the elements of $G(m,1,n)$ with respect to $G(m,1,n-1)$.

Proposition 34. *Any element $x \in G(m,1,n)$ can be written in the form :*

$$x = s_j s_{j-1} \dots s_1 t^\alpha s_1 s_2 \dots s_{n-1} w \ , \tag{A.1}$$

where $j \in \{0,\dots,n-1\}$, $\alpha \in \{0,\dots,m-1\}$ and $w \in W \simeq G(m,1,n-1)$. Here the standard notation is employed: the empty product, e.g. for $j=0$ in (A.1), is 1.

Given a normal form for elements of $G(m,1,n-1)$, the Proposition 34 provides a normal form for elements of $G(m,1,n)$. In particular, we can use the same (with n replaced by $n-1$), as in the Proposition 34, normal form for elements of $G(m,1,n-1)$ constructing recursively the global normal form for elements of $G(m,1,n)$.

A.2 Normal form. Preparation

The normal form for elements of the cyclotomic Hecke algebra $H(m,1,n)$ we shall construct in several steps.

1. Normal form for elements of $H(m,1,n)$; beginning of the construction. The normal form from the Proposition 34 generalizes to the cyclotomic Hecke algebra $H(m,1,n)$ generated by $\tau, \sigma_1, \dots, \sigma_{n-1}$ with the defining relations (2.1)–(2.4) and (2.7)–(2.8). We start with the following Proposition.

Proposition 35. *Any $x \in H(m,1,n)$ can be written as a linear combination of elements*

$$\sigma_j^{-1}\sigma_{j-1}^{-1}\dots\sigma_1^{-1}\tau^\alpha\sigma_1\sigma_2\dots\sigma_{n-1}\tilde{w} \ , \tag{A.2}$$

where $j \in \{0,\dots,n-1\}$, $\alpha \in \{0,\dots,m-1\}$ and $\tilde{w} \in \tilde{W}$ with $\tilde{W}$ the subalgebra generated by the elements $\tau, \sigma_1, \dots, \sigma_{n-2}$.

[4] The induction starts with the group $G(m,1,1)$ generated by τ only; clearly, $|G(m,1,1)| = m$.

Proof. The unit element of the algebra $H(m,1,n)$ is in the set of elements (A.2), as well as all generators. We only have to check that the linear span of the elements (A.2) is stable under multiplication by the generators (then the linear span of the elements (A.2) is a unital subalgebra, containing all generators, thus the whole algebra). We consider the left multiplication; we multiply an arbitrary element E of the form (A.2) by each generator $\mathsf{G} \in \{\sigma_1, \dots, \sigma_{n-1}, \tau\}$ from the left and move in this product GE the original generator G to the right; the expression transforms; we follow the process until it becomes clear that the original product GE is a linear combination of elements of the form (A.2).

(i) We multiply the element (A.2) from the left by σ_i with $i > j+1$. The element σ_i commutes with $\sigma_j^{-1}\sigma_{j-1}^{-1} \dots \sigma_1^{-1}\tau^\alpha$ and thus moves through the combination $\sigma_j^{-1}\sigma_{j-1}^{-1} \dots \sigma_1^{-1}\tau^\alpha$ to the right without changes; then σ_i moves through $\sigma_1\sigma_2 \dots \sigma_{n-1}$ to the right, becoming σ_{i-1}; $\sigma_{i-1}\tilde{w}$ is again in $\tilde{W}$ and we are done.

(ii) We multiply the element (A.2) from the left by σ_i with $i < j$. When σ_i moves through $\sigma_j^{-1}\sigma_{j-1}^{-1} \dots \sigma_1^{-1}$ it transforms into σ_{i+1}; the element σ_{i+1} commutes through τ^α to the right without changes and then σ_{i+1} moves through $\sigma_1\sigma_2 \dots \sigma_{n-1}$ to the right, becoming σ_i; as in (i), $\sigma_i\tilde{w} \in \tilde{W}$.

(iii) The assertion is immediate when we multiply the element (A.2) from the left by σ_j.

(iv) When we multiply the element (A.2) from the left by σ_{j+1}, for the proof it is enough to write $\sigma_{j+1} = \sigma_{j+1}^{-1} + (q - q^{-1})$.

(v) We multiply the element (A.2) from the left by τ. If $j = 0$, there is nothing to do. Let $j > 0$. The element τ moves to the right until it reaches σ_1^{-1}. Then we use the Lemma 36 below and obtain three terms. For the first term:

$$\begin{aligned}\sigma_j^{-1} \dots \sigma_2^{-1} \cdot \tau\sigma_1\tau^\alpha \cdot \sigma_2 \dots \sigma_{n-1} \cdot \tilde{w} \;&= \sigma_j^{-1} \dots \sigma_2^{-1} \cdot \tau \cdot \sigma_1\sigma_2 \dots \sigma_{n-1} \cdot \tau^\alpha\tilde{w} \\ &= \tau \cdot \sigma_1\sigma_2 \dots \sigma_{n-1} \cdot \sigma_{j-1}^{-1} \dots \sigma_1^{-1}\tau^\alpha\tilde{w}\end{aligned}$$

and $\sigma_{j-1}^{-1} \dots \sigma_1^{-1}\tau^\alpha\tilde{w} \in \tilde{W}$. For the second term:

$$\sigma_j^{-1} \dots \sigma_2^{-1} \cdot \tau^{\alpha+1}\sigma_1\sigma_2 \dots \sigma_{n-1} \cdot \tilde{w} = \tau^{\alpha+1}\sigma_1\sigma_2 \dots \sigma_{n-1} \cdot \sigma_{j-1}^{-1} \dots \sigma_1^{-1}\tilde{w}$$

and $\sigma_{j-1}^{-1} \dots \sigma_1^{-1}\tilde{w} \in \tilde{W}$. For the third term:

$$\sigma_j^{-1} \dots \sigma_1^{-1} \cdot \tau^\alpha\sigma_1\tau\sigma_2 \dots \sigma_{n-1} \cdot \tilde{w} = \sigma_j^{-1} \dots \sigma_1^{-1} \cdot \tau^\alpha\sigma_1\sigma_2 \dots \sigma_{n-1} \cdot \tau\tilde{w}$$

and $\tau\tilde{w} \in \tilde{W}$. $\square$

Lemma 36. *For $\alpha \geq 0$ we have:*

$$\tau\sigma_1^{-1}\tau^\alpha\sigma_1 = (q - q^{-1})\big(\tau\sigma_1\tau^\alpha - \tau^{\alpha+1}\sigma_1\big) + \sigma_1^{-1}\tau^\alpha\sigma_1\tau \ . \tag{A.3}$$

Proof of the Lemma. Multiplying the equality $\sigma_1\tau\sigma_1\tau^\alpha = \tau^\alpha\sigma_1\tau\sigma_1$ by σ_1^{-1} from both sides, we get

$$\tau\sigma_1\tau^\alpha\sigma_1^{-1} = \sigma_1^{-1}\tau^\alpha\sigma_1\tau \ . \tag{A.4}$$

Expanding $\tau\sigma_1^{-1}\tau^\alpha\sigma_1 = \tau\big(\sigma_1 - (q - q^{-1})\big)\tau^\alpha\big(\sigma_1^{-1} + (q - q^{-1})\big)$ and using (A.4) we obtain (A.3). $\square$

2. Normal form for elements of $H(m,1,n)$; continuation. Consider an arbitrary $H(m,1,n-1)$-module M_{n-1}. We denote its elements by letters u, v etc. Let $E = \mathbb{C}[z]/<\chi>$ where $\chi(z)$ is the characteristic polynomial for τ. Let V be a vector space with the basis v_j, $j = 0, \dots, n-1$. Let $M_n := V \otimes E \otimes M_{n-1}$. The elements $v_j \otimes \phi \otimes u$ we denote by $\mathcal{V}_{j,\phi,u}$. For brevity we write

$$\beta_j := \sigma_{j-1}^{-1} \dots \sigma_1^{-1} \ ;$$

the result of the action of an element $\psi \in H(m,1,n-1)$ on an element $u \in M_{n-1}$ we denote simply by ψu (without any symbol for the action).

Define operators F_{σ_i}, $i = 1, \dots, n-1$, and F_τ on the space M_n by (below the last index of $\mathcal{V}$ carries information about the action of the elements of the algebra $H(m,1,n-1)$ on the module M_{n-1}; ϕ stands for an element of E, a polynomial in z, defined modulo $\chi(z)$; the element $\phi(\tau) \in H(m,1,n-1)$ which appears in the last index of $\mathcal{V}$ is therefore well defined):

$$F_{\sigma_i} \ : \mathcal{V}_{j,\phi,u} \ \mapsto \ \begin{cases} \mathcal{V}_{j,\phi,\sigma_{i-1}u} \ , & j < i-1 \ , \\ (q - q^{-1})\, \mathcal{V}_{i-1,\phi,u} + \mathcal{V}_{i,\phi,u} \ , & j = i-1 \ , \\ \mathcal{V}_{i-1,\phi,u} \ , & j = i \ , \\ \mathcal{V}_{j,\phi,\sigma_i u} \ , & j > i \ , \end{cases} \tag{A.5}$$

and

$$F_\tau \ : \begin{cases} \mathcal{V}_{0,\phi,u} \ \mapsto \ \mathcal{V}_{0,z\phi,u} \ , \\ \mathcal{V}_{j,\phi,u} \ \mapsto \ (q-q^{-1})\, \mathcal{V}_{0,z,\beta_j\phi(\tau)u} - (q-q^{-1})\, \mathcal{V}_{0,z\phi,\beta_j u} + \mathcal{V}_{j,\phi,\tau u} \ , \quad j > 0 \ . \end{cases} \tag{A.6}$$

Let, as above, $\tilde{W}$ denote the subalgebra of the algebra $H(m,1,n)$ generated by the elements τ and $\sigma_1, \dots, \sigma_{n-2}$. Take $\tilde{W}$ for the $H(m,1,n-1)$-module M_{n-1} (in general, the algebra $\tilde{W}$ is a quotient of $H(m,1,n-1)$; we define the action of $H(m,1,n-1)$ by the left multiplication on its quotient). By a direct calculation one checks that the formulas (A.5) and (A.6) are valid if one substitutes $\sigma_j^{-1}\sigma_{j-1}^{-1} \dots \sigma_1^{-1}\phi(\tau)\sigma_1\sigma_2 \dots \sigma_{n-1}u$, as in (A.2), for $\mathcal{V}_{j,\phi,u}$. More is true. The formulas (A.5) and (A.6) have the following universal property.

Proposition 37. *The map $\sigma_i \mapsto F_{\sigma_i}$, $i = 1, \dots, n-1$, and $\tau \mapsto F_\tau$ equips M_n with a structure of a $H(m,1,n)$-module.*

Proof. A straightforward although lengthy calculation. Given a defining relation from the list (2.1)–(2.4) and (2.7)–(2.8) we verify it on each vector $\mathcal{V}_{j,\phi,u}$. Below we mention different placements of the index j in a verification of a given relation.

(i) For the relation $\sigma_i\sigma_k = \sigma_k\sigma_i$ with $i < k-1$ one considers separately the following positions of the index j:

$$j < i-1 \ , \ j = i-1 \ , \ j = i \ , \ i < j < k-1 \ , \ j = k-1 \ , \ j = k \ \text{ and } \ j > k \ .$$

(ii) For the Artin relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ one considers separately the following positions of the index j:

$$j < i-1 \ , \ j = i-1 \ , \ j = i \ , \ j = i+1 \ \text{ and } \ j > i+1 \ .$$

(iii) For the relation $\sigma_i^2 - (q-q^{-1})\sigma_i - 1 = 0$ one considers separately the following positions of the index j:

$$j < i-1 \ , \ j = i-1 \ , \ j = i \ \text{ and } \ j > i \ .$$

(iv) For the relation $\tau\sigma_i = \sigma_i\tau$ with $i > 1$ one considers separately the following positions of the index j:

$$j = 0 \ , \ j < i-1 \ , \ j = i-1 \ , \ j = i \ \text{ and } \ j > i \ .$$

(v) For the relation $\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$ it is enough to consider separately the following positions of the index j:

$$j = 0 \ , \ j = 1 \ \text{ and } \ j > 1 \ .$$

The following observation simplifies a verification here:

$$F_\tau F_{\sigma_1} F_\tau : \begin{cases} \mathcal{V}_{0,\phi,u} \mapsto (q-q^{-1})\,\mathcal{V}_{0,z,\tau\phi(\tau)u} + \mathcal{V}_{1,z\phi,\tau u} \ , \\[2ex] \mathcal{V}_{1,\phi,u} \mapsto (q-q^{-1})\,\mathcal{V}_{1,z,\tau\phi(\tau)u} - (q-q^{-1})\,\mathcal{V}_{1,z\phi,\tau u} + \mathcal{V}_{0,z\phi,\tau u} \ , \\[2ex] \mathcal{V}_{j,\phi,u} \mapsto (q-q^{-1})^2\,\mathcal{V}_{0,z,\tau[\beta_j,\phi(\tau)]u} + (q-q^{-1})\,\mathcal{V}_{1,z,\tau\beta_j\phi(\tau)u} - (q-q^{-1})\,\mathcal{V}_{1,z\phi,\tau\beta_j u} \\[1ex] \qquad\quad +(q-q^{-1})\,\mathcal{V}_{0,z,\beta_j\phi(\tau)\sigma_1\tau u} - (q-q^{-1})\,\mathcal{V}_{0,z\phi,\beta_j\sigma_1\tau u} + \mathcal{V}_{j,\phi,\tau\sigma_1\tau u} \ , \quad j > 1 \ . \end{cases} \tag{A.7}$$

Here $[\beta_j, \phi(\tau)]$ is the commutator of β_j and $\phi(\tau)$; in the verification of the relation $\tau\sigma_1\tau\sigma_1 = \sigma_1\tau\sigma_1\tau$ on $\mathcal{V}_{j,\phi,u}$ with $j > 1$ we use the formula (A.3) in the form

$$\tau\sigma_1^{-1}\phi(\tau)\sigma_1 = (q-q^{-1})\big(\tau\sigma_1\phi(\tau) - \tau\phi(\tau)\sigma_1\big) + \sigma_1^{-1}\phi(\tau)\sigma_1\tau \ .$$

(vi) For the relation $(\tau - v_1)\dots(\tau - v_m) = 0$ one considers separately the following positions of the index j:

$$j = 0 \ \text{ and } \ j > 0 \ .$$

A verification of this relation for $j > 0$ is the only place in the proof which maybe requires a comment. For $j > 0$ one proves by induction the following formula:

$$F_\tau^l \ : \mathcal{V}_{j,\phi,u} \mapsto (q-q^{-1}) \sum_{i=1}^{l} \mathcal{V}_{0,z^i,\beta_j\phi(\tau)\tau^{l-i}u} - (q-q^{-1}) \sum_{i=1}^{l} \mathcal{V}_{0,z^i\phi,\beta_j\tau^{l-i}u} + \mathcal{V}_{j,\phi,\tau^l u} \ , \quad j > 0 \ . \tag{A.8}$$

The first sum in (A.8) can be seen as the image of an element

$$(z\otimes\phi)\cdot\frac{1\otimes z^l - z^l\otimes 1}{1\otimes z - z\otimes 1} \tag{A.9}$$

from the space $E \otimes E$ to M_n with respect to the map κ_u, defined for each $u \in M_{n-1}$ by

$$\kappa_u \; : \; f \otimes g \mapsto \mathcal{V}_{0,f,\beta_j g(\tau) u} \ .$$

In (A.9), the fraction $\frac{1 \otimes z^l - z^l \otimes 1}{1 \otimes z - z \otimes 1}$ is understood as the image of a polynomial which is the result of the division of the numerator by the denominator (as polynomials of two unrestricted variables) in the space $E \otimes E$. Similarly, the second sum in (A.8) can be understood as the image of $(z\phi \otimes 1) \cdot \frac{1 \otimes z^l - z^l \otimes 1}{1 \otimes z - z \otimes 1}$ with respect to the same map κ_u. Thus the first sum minus the second sum (the combination which appears in (A.8)) is the image of

$$(1 \otimes \phi - \phi \otimes 1) \cdot (z \otimes 1) \cdot \frac{1 \otimes z^l - z^l \otimes 1}{1 \otimes z - z \otimes 1} \ . \tag{A.10}$$

The element (A.10) already as a polynomial (and therefore as an element in $E \otimes E$) can be written in the form

$$\frac{1 \otimes \phi - \phi \otimes 1}{1 \otimes z - z \otimes 1} \cdot (z \otimes 1) \cdot (1 \otimes z^l - z^l \otimes 1) \ , \tag{A.11}$$

where the fraction $\frac{1 \otimes \phi - \phi \otimes 1}{1 \otimes z - z \otimes 1}$ is understood again as the image of a polynomial which is the result of the division of the numerator by the denominator (as polynomials of two unrestricted variables) in the space $E \otimes E$.

Writing now $\chi(z) = \sum_{l=0}^{m} c_m z^m$ one verifies the relation $\chi(F_\tau) = 0$ immediately with the help of (A.11) (recall that $E = \mathbb{C}[z]/ < \chi >$). $\square$

Remark. The action of F_τ on the vectors of the form $\mathcal{V}_{j,1,u}$ with $j > 0$ is simply the action of τ on M_{n-1}, that is,

$$F_\tau \ : \ \mathcal{V}_{j,1,u} \ \mapsto \ \mathcal{V}_{j,1,\tau u} \quad \text{for} \ \ j > 0 \ . \tag{A.12}$$

Remark. The operators F_{σ_i} and F_τ defined in (A.5) and (A.6) can be represented by $n \times n$ matrices (with indices related to the space V) whose elements are operators acting in the space $E \otimes M_{n-1}$. By $\hat{z}$ we denote the operator of the multiplication by z in the space E. To fit formulas in the line, we denote the operator $\mathrm{Id}_E \otimes \sigma_i$ simply by σ_i, the operator $\mathrm{Id}_E \otimes \tau$ simply by τ and the operator $\hat{z} \otimes \mathrm{Id}_{M_{n-1}}$ simply by $\hat{z}$.

The operator F_{σ_i} reads (recall that the elements of the basis of the space V are labeled by numbers from 0 to $n-1$)

$$F_{\sigma_i} = \begin{pmatrix} \sigma_{i-1} & & & & & & \\ & \ddots & & & & & \\ & & \sigma_{i-1} & & & & \\ & & & q - q^{-1} & 1 & & \\ & & & 1 & 0 & & \\ & & & & & \sigma_i & \\ & & & & & & \ddots \end{pmatrix} \ ; \tag{A.13}$$

here the 2×2 block with ones (that is, the identity operators) outside the main diagonal is in the $(i-1)^{st}$ and i^{th} lines and columns.

The operator F_τ reads

$$F_\tau = \begin{pmatrix} \hat{z} & (q-q^{-1})\hat{z}(\mu-1) & (q-q^{-1})\hat{z}\sigma_1^{-1}(\mu-1) & (q-q^{-1})\hat{z}\sigma_2^{-1}\sigma_1^{-1}(\mu-1) & \dots \\ & \tau & & & \dots \\ & & \tau & & \dots \\ & & & \tau & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} ; \qquad \text{(A.14)}$$

here only the first line and the main diagonal have non-zero entries. The operator μ on the space $E \otimes M_{n-1}$ is defined as follows:

$$\mu(\phi \otimes u) := 1 \otimes \phi(\tau)u \ ,$$

where ϕ is a polynomial in z. The operator μ has the following properties:

$$\mu\hat{z} = \tau\mu \ , \ \mu\tau = \tau\mu \ , \ \mu^2 = \mu \ .$$

A.3 Flatness of deformation. Normal form for elements of $H(m,1,n)$

We are now ready to prove that the deformation $H(m,1,n)$ of the group ring $\mathbb{C}G(m,1,n)$ is flat and to give the normal form for the elements of the algebra $H(m,1,n)$.

1. Flatness of deformation.

As above, $\tilde{W}$ denotes the subalgebra of the algebra $H(m,1,n)$ generated by the elements τ and $\sigma_1, \dots, \sigma_{n-2}$.

Proposition 38. *(i) The algebra $H(m,1,n)$ is a flat deformation of the group ring $\mathbb{C}G(m,1,n)$; in other words, $H(m,1,n)$ is a free $\mathbb{C}[q,q^{-1},v_1,\dots,v_m]$-module of dimension*

$$\dim\big(H(m,1,n)\big) = |G(m,1,n)| = n!m^n \ . \qquad \text{(A.15)}$$

(ii) Moreover the subalgebra $\tilde{W}$ is isomorphic to $H(m,1,n-1)$.

Proof. We may assume, by induction, that

$$\dim\big(H(m,1,n-1)\big) = (n-1)! \cdot m^{n-1} \ . \qquad \text{(A.16)}$$

The induction starts with the algebra $H(m,1,1)$ (with only one generator τ) for which the formula $\dim\big(H(m,1,1)\big) = m$ clearly holds.

The Proposition 35 implies that

$$\dim\big(H(m,1,n)\big) \leq mn\dim(\tilde{W}) \ . \tag{A.17}$$

In general $\tilde{W}$ is a quotient of $H(m,1,n-1)$. Thus, by (A.16),

$$\dim\big(H(m,1,n)\big) \leq n!m^n \ . \tag{A.18}$$

Let M_{n-1} be a left regular module for $H(m,1,n-1)$; that is, the space of the module is the algebra itself and the elements of the algebra act by left multiplication.

By (A.16), $M_n = V \otimes E \otimes M_{n-1}$ is a vector space of dimension $n \cdot m \cdot (n-1)! \cdot m^{n-1} = n!m^n$. The vector space M_n has a structure of an $H(m,1,n)$-module given by the Proposition 37.

We denote by F_a the operator corresponding to an element a. Using the formulas (A.5) and (A.6) for the action of $H(m,1,n)$ on M_n and also the formula (A.12), we have

$$F_{\sigma_j^{-1}\dots\sigma_1^{-1}\phi(\tau)\sigma_1\dots\sigma_{n-1}u} \ : \ \mathcal{V}_{n-1,1,1} \ \mapsto \ \mathcal{V}_{j,\phi,u} \ . \tag{A.19}$$

Choosing an arbitrary basis $\{u_c\}$, $c=1,\dots,(n-1)!\cdot m^{n-1}$, of $H(m,1,n-1)$, we see that the operators $F_{\sigma_j^{-1}\dots\sigma_1^{-1}\phi(\tau)\sigma_1\dots\sigma_{n-1}u_c}$ are independent. We deduce that

$$\dim\big(H(m,1,n)\big) \geq n!m^n \ . \tag{A.20}$$

The combination of (A.18) and (A.20) implies the statement (i) of the Proposition.

If $\tilde{W}$ is a non-trivial quotient of $H(m,1,n-1)$ then $\dim(\tilde{W}) < (n-1)! \cdot m^{n-1}$ and, by (A.17), $\dim\big(H(m,1,n)\big) < n!m^n$, contradicting to the already established statement (i); the assertion (ii) follows completing the proof of the Proposition. □

2. Normal form. We can, in the same way as we did for $G(m,1,n)$, construct recursively a global normal form for elements of $H(m,1,n)$ using now the Proposition 35 and the Proposition 38, statement (ii). Let R_k be the set of elements $\{\sigma_j^{-1}\sigma_{j-1}^{-1}\dots\sigma_1^{-1}\tau^\alpha\sigma_1\sigma_2\dots\sigma_{k-1},\ j=0,\dots,k-1,\ \alpha=0,\dots,m-1\}$.

Corollary 39. *Any element $x \in H(m,1,n)$ can be written uniquely as a linear combination of elements*

$$x = u_n u_{n-1}\dots u_1 \ , \tag{A.21}$$

where $u_k \in R_k$ for $k=1,\dots,n$.

In other words, the products $u_n u_{n-1}\dots u_1$, where u_k ranges over R_k for $k=1,\dots,n$, form a basis of the vector space $H(m,1,n)$.

Remark. Define the homomorphism $\varsigma : H_n \to H(m,1,n)$ from the Hecke algebra H_n to the cyclotomic Hecke algebra $H(m,1,n)$ by sending the generator σ_i of H_n to the generator σ_i of $H(m,1,n)$ for $i = 1,\dots,n-1$. Words (A.21) in which τ does not enter are contained in the subalgebra of $H(m,1,n)$ generated by σ_i, $i = 1,\dots,n-1$ and therefore in the image of the algebra H_n under the map ς. The cardinality of the set of such words is $n!$. Since $\dim(H_n) = n!$, the uniqueness from the Corollary 39 implies that ς is an embedding.

There is another way, without the use of the Corollary 39, to check that ς is an embedding. Fix a number e, $1 \le e \le m$. The map which sends the generator σ_i of $H(m,1,n)$ to the generator σ_i of H_n for $i = 1,\dots,n-1$ and the generator τ of $H(m,1,n)$ to the number v_e clearly extends to a homomorphism $\pi_e : H(m,1,n) \to H_n$. One has $\pi_e \circ \varsigma = \mathrm{Id}_{H_n}$ so π_e is a left inverse to ς (for each e); in particular, ς is an embedding.

The maps used in the last argument play the same role as the maps $\hat{\iota}$ and π, see (6.13) and (6.14), for the embedding $\mathbb{C}G(m,1,n) \to \mathfrak{A}_{m,n}$. Likewise, we have embeddings $H_n \to \hat{H}_n$ and, on the level of groups, $B_n \to \alpha B_n$ defined each time by a map, tautological on generators.

3. Induced representations. Let $\mathfrak{B}$ be an associative subalgebra of an associative algebra $\mathfrak{A}$. Let W be a left $\mathfrak{B}$-module. The vector space $\mathfrak{A} \otimes_{\mathfrak{B}} W$ carries a natural $\mathfrak{A}$-module structure defined by $\mathfrak{a}.(\mathfrak{a}' \otimes w) := \mathfrak{a}\mathfrak{a}' \otimes w$. This is the induced $\mathfrak{A}$-module.

The Proposition 38 (or the Corollary 39) implies the uniqueness of the form (A.2) for the elements of $H(m,1,n)$. Taking into account the Proposition 37 we arrive at the following conclusion.

Corollary 40. *The module M_n is the induced $H(m,1,n)$-module (with respect to the subalgebra $H(m,1,n-1)$ and the module M_{n-1} over it). The formulas (A.5) and (A.6) give an explicit realization of the induced module M_n.*

4. Comments on formulas (A.5) and (A.6).

(a) With the help of the formulas (A.5) and (A.6) we have constructed the $H(m,1,n)$-module structure on the space $V \otimes E \otimes M_{n-1}$, where M_{n-1} is an arbitrary $H(m-1,1,n)$-module. In the particular situation when m equals 1 the space E is one-dimensional and we can canonically identify it with the field $\mathbb{C}$. As a result we obtain the operators F_{σ_i} acting on the space $V \otimes M_{n-1}$:

$$F_{\sigma_i} \ : \mathcal{V}_{j,u} \ \mapsto \ \begin{cases} \mathcal{V}_{j,\sigma_{i-1}u} \ , & j < i-1 \ , \\ (q - q^{-1})\, \mathcal{V}_{i-1,u} + \mathcal{V}_{i,u} \ , & j = i-1 \ , \\ \mathcal{V}_{i-1,u} \ , & j = i \ , \\ \mathcal{V}_{j,\sigma_i u} \ , & j > i \ . \end{cases} \tag{A.22}$$

The formula (A.22) constructs, now on the vector space $M_n := V \otimes M_{n-1}$, a module over the Hecke algebra H_n: the operator F_τ was not used in verifying the Hecke algebra relations (that is, relations (2.1)–(2.2) and (2.7)) for the operators F_{σ_i} (we leave the check of it as an exercise for the reader) in the proof of the Proposition 37.

Another way to arrive at (A.22) is to notice that in the formula (A.5) alone, the label ϕ of vectors $\mathcal{V}_{j,\phi,u}$ is not touched by the action of the operators F_{σ_i}, so one may omit the label ϕ and reproduce the formula (A.22).

(a′) In particular, taking for M_{n-1} the one-dimensional module of the Hecke algebra H_{n-1}, in which the generators σ_i are mapped to q, the resulting module M_n is the Burau module for the Hecke algebra H_n.

(b) In verifying relations (2.3)–(2.4) for the operators F_{σ_i} and F_τ given by the formulas (A.5) and (A.6) we used the characteristic equation for σ but we did not use the characteristic equation for τ (we leave the check of it as an exercise for the reader). So in fact the formulas (A.5) and (A.6) define a module over the affine Hecke algebra[5] $\hat{H}_n$ starting from a module over the affine Hecke algebra $\hat{H}_{n-1}$. Note that $E = \mathbb{C}[z]$ in this situation.

(b′) Taking now for M_{n-1} the one-dimensional module of the cyclotomic Hecke algebra $H_{m,1,n-1}$ (respectively, the affine Hecke algebra $\hat{H}_{n-1}$), in which the generators σ_i are mapped to q and the generator τ is mapped to v_e for some e, $1 \leq e \leq m$, (v_e is arbitrary for the affine Hecke algebra), the resulting $H_{m,1,n}$-module (respectively, $\hat{H}_n$-module) on the space $M_n = V \otimes E$ is the natural analogue of the Burau module (with $E = \mathbb{C}[z]$ in the affine Hecke algebra situation) . The action of the generators σ_i is given by the usual Burau matrices (these are the matrices (A.13) in which every σ is replaced by q; these matrices act trivially in the space E) while the matrix of the operator τ is given by

$$F_\tau = \begin{pmatrix} \hat{z} & (q-q^{-1})\hat{z}(\mu_e - 1) & (q-q^{-1})q^{-1}\hat{z}(\mu_e - 1) & (q-q^{-1})q^{-2}\hat{z}(\mu_e - 1) & \dots \\ & v_e & & & \dots \\ & & v_e & & \dots \\ & & & v_e & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} ; \tag{A.23}$$

here μ_e is defined by $\mu_e(\phi) := \phi(v_e)$, where ϕ is a polynomial in z.

(c) Naturality.

For a vector space M let $\Upsilon_A(M) := V \otimes M$ and $\Upsilon_B(M) := V \otimes E \otimes M$. Here V is a vector space with the fixed basis v_j, $j = 0, \dots, n-1$. The space E will be specified below. For a map $\alpha : M \to M'$ let $\Upsilon_A(\alpha) := \mathrm{Id}_V \otimes \alpha$ and $\Upsilon_B(\alpha) := \mathrm{Id}_V \otimes \mathrm{Id}_E \otimes \alpha$.

The constructions (A.5)–(A.6) and (A.22) possess the following functoriality properties. This follows from the observation stating that the constructions (A.5)–(A.6) and (A.22) provide a realizations

[5]We recall that the affine Hecke algebra $\hat{H}_n$ is the algebra generated by the elements $\sigma_1, \dots, \sigma_{n-1}$ and τ with the defining relations (2.1)–(2.4) and (2.7).

of induced representations (see the Corollary 40 for the cyclotomic algebras $H(m,1,n)$; the assertion about the induced representations can be extended to the other two cases listed below); these functoriality properties can be checked directly as well.

- Υ_B is a functor from the category of $H_{m,1,n-1}$-modules to the category of $H_{m,1,n}$-modules,
$$\Upsilon_B : H_{m,1,n-1}\text{-mod} \ \rightarrow \ H_{m,1,n}\text{-mod} \ ;$$
here $E = \mathbb{C}[z]/ < \chi >$ where $\chi(z)$ is the characteristic polynomial for τ.
- In particular, for $m=1$, Υ_A is a functor from the category of H_{n-1}-modules to the category of H_n-modules,
$$\Upsilon_A : H_{n-1}\text{-mod} \ \rightarrow \ H_n\text{-mod} \ .$$
- Also, Υ_B is a functor on the level of the affine Hecke algebras
$$\Upsilon_B : \hat{H}_{n-1}\text{-mod} \ \rightarrow \ \hat{H}_n\text{-mod} \ .$$
Here $E = \mathbb{C}[z]$.

The classical limit does not cause any difficulties: one simply sets q to 1 in the formulas (A.5)–(A.6) and (A.22) (note that the parameters v_e participate in the formulas (A.5)–(A.6) only through the action of τ on M_{n-1}).

5. Other normal forms. There exist three other normal forms for elements of $H(m,1,n)$ with respect to $\tilde{W}$ similar to the form from the Proposition 35. Any $x \in H(m,1,n)$ can be written as a linear combination of elements of the set

$$\begin{cases} \sigma_{k+1}\sigma_{k+2}\dots\sigma_{n-1}\tilde{w} \ , \ \text{where } k\in\{0,\dots,n-1\} \text{ and } \tilde{w}\in\tilde{W} \ , \\[2ex] \sigma_j\sigma_{j-1}\dots\sigma_1\tau^\alpha\sigma_1\dots\sigma_{n-1}\tilde{w} \ , \ \text{where } j\in\{0,\dots,n-1\},\ \alpha\in\{1,\dots,m-1\} \text{ and } \tilde{w}\in\tilde{W} \ , \end{cases} \tag{A.24}$$

or of the set

$$\sigma_j\sigma_{j-1}\dots\sigma_1\tau^\alpha\sigma_1^{-1}\dots\sigma_{n-1}^{-1}\tilde{w} \ , \ \text{where } j\in\{0,\dots,n-1\},\ \alpha\in\{0,\dots,m-1\} \text{ and } \tilde{w}\in\tilde{W} \ , \tag{A.25}$$

or of the set

$$\begin{cases} \sigma_{k+1}^{-1}\sigma_{k+2}^{-1}\dots\sigma_{n-1}^{-1}\tilde{w} \ , \ \text{where } k\in\{0,\dots,n-1\} \text{ and } \tilde{w}\in\tilde{W} \ , \\ \sigma_j^{-1}\sigma_{j-1}^{-1}\dots\sigma_1^{-1}\tau^\alpha\sigma_1^{-1}\dots\sigma_{n-1}^{-1}\tilde{w} \ , \ \text{where } j\in\{0,\dots,n-1\},\ \alpha\in\{1,\dots,m-1\} \text{ and } \tilde{w}\in\tilde{W} \ , \end{cases} \tag{A.26}$$

where $\tilde{W}$ is, as above, the subalgebra generated by $\tau,\sigma_1,\dots,\sigma_{n-2}$.

Concerning the normal form (A.24), the proof goes along the same lines as the proof of the Proposition 35.

The forms (A.25) and (A.26) can be reduced to the normal forms (A.2) and (A.24) by applying standard automorphisms of the cyclotomic Hecke algebras.

Appendix B. Bratteli diagrams and their products

Here we recall several facts about Bratteli diagrams (see, e.g., [10]) and their graded products. Then we recall the information, needed in the main body of the text, about the dimensions of the vertices in the powers of the Young graph.

B.2 Bratteli diagrams

A Bratteli diagram is a graded graph; this means that there is a function, called *degree*, from the set of vertices to the set of non-negative integers. There is only one degree 0 vertex which is denoted by $\varnothing$. The edges of the graph can only connect two vertices with neighboring degrees ("neighboring" means that the absolute value of the difference of the degrees is 1). When one draws a Bratteli diagram it is convenient to place on the level a all the vertices of degree a (the level a stands for the value of the ordinate $(-a)$). Then a vertex on the level a has "incoming" edges from the level $a-1$ and "outgoing" edges to the level $a+1$. The dimension of a vertex x is the number of paths which go down from the vertex $\varnothing$ to x.

In the representation theory, one associates a Bratteli diagram to an ascending chain

$$\mathcal{A}_0 = \mathbb{C} \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \dots \tag{B.1}$$

of associative algebras: vertices of degree k correspond to representations (depending on circumstances, indecomposable, irreducible *etc.*) of the algebra $\mathcal{A}_k$ and the Bratteli diagram visualizes the branching rules for the pairs $(\mathcal{A}_{k+1}, \mathcal{A}_k)$, $k \in \mathbb{Z}$. In this situation, the dimension of a vertex is simply equal to the dimension of the corresponding representation.

Let $\mathfrak{G}^{(1)}$ and $\mathfrak{G}^{(2)}$ be two Bratteli diagrams. The vertices of the product $\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)}$ are by definition couples (x, y) where x is a vertex of $\mathfrak{G}^{(1)}$ and y is a vertex of $\mathfrak{G}^{(2)}$. The degree of (x, y) is the sum of the degree of x and the degree of y. The top vertex $\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)}$ which is denoted again by $\varnothing$ is therefore $(\varnothing, \varnothing)$. If there is an edge between x and x' in $\mathfrak{G}^{(1)}$ one draws an edge between (x, y) and (x', y) for all y; we say that these edges are of type 1. If there is an edge between y and y' in $\mathfrak{G}^{(2)}$ one draws an edge between (x, y) and (x, y') for all x; we say that these edges are of type 2. By definition these are all edges: each edge is either of type 1 or of type 2.

Iterating, we define the product of an arbitrary number m of Bratteli diagrams.

B.2 Dimensions of vertices of the product

Let $\mathfrak{G}$ be a Bratteli diagram. Denote by $\mathfrak{P}(\mathfrak{G})$ the set of paths which begin at the top vertex of $\mathfrak{G}$ and go down. For $p \in \mathfrak{P}(\mathfrak{G})$ denote by $\mathcal{E}(p)$ the collection of edges of p and by $\mathrm{end}(p)$ the end point of p; if $z = \mathrm{end}(p)$ then $\deg(z)$ equals the length of p, the cardinality of $\mathcal{E}(p)$. The set $\mathcal{E}(p)$ is naturally ordered: the steps in the path follow one after another.

Let (x, y) be a vertex of $\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)}$. Let p be a path from $\mathfrak{P}(\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)})$ with $\mathrm{end}(p) = (x, y)$. The set $\mathcal{E}(p)$ is the disjoint union of two subsets, $\mathcal{E}_1(p)$ and $\mathcal{E}_2(p)$, where $\mathcal{E}_1(p)$ (respectively, $\mathcal{E}_2(p)$) is the subset of $\mathcal{E}(p)$ consisting of edges of type 1 (respectively, type 2). Each edge from $\mathcal{E}_1(p)$ naturally defines an edge in the graph $\mathfrak{G}^{(1)}$ and the set of edges thus defined form a path p_1 in the graph $\mathfrak{G}^{(1)}$ going down from $\varnothing$ of $\mathfrak{G}^{(1)}$ to x, $p_1 \in \mathfrak{P}(\mathfrak{G}^{(1)})$; similarly, each edge from $\mathcal{E}_2(p)$ naturally defines an edge in the graph $\mathfrak{G}^{(2)}$ and the set of edges thus defined form a path p_2 in the graph $\mathfrak{G}^{(2)}$ going down from $\varnothing$ of $\mathfrak{G}^{(2)}$ to y, $p_2 \in \mathfrak{P}(\mathfrak{G}^{(2)})$. We have therefore a map from $\mathfrak{P}(\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)})$ to the product $\mathfrak{P}(\mathfrak{G}^{(1)}) \times \mathfrak{P}(\mathfrak{G}^{(2)})$, defined by

$$\pi : p \mapsto (p_1, p_2) \ . \tag{B.2}$$

One cannot reconstruct uniquely the path p knowing the paths p_1 and p_2. Let a be the degree of x and b the degree of y. Any order $\succ$ on the union $\mathcal{E}(p_1) \cup \mathcal{E}(p_2)$ which is compatible with the natural orders on $\mathcal{E}_1(p)$ and $\mathcal{E}_2(p)$ (in the sense that if a step γ is after a step γ' in $\mathcal{E}_1(p)$ or $\mathcal{E}_2(p)$ then γ is after γ' with respect to the order $\succ$) determines a path from $\mathfrak{P}(\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)})$ In other words, in the sequence of $a+b$ edges of a path of length $a+b$ from $\mathfrak{P}(\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)})$ one can assign the type 1 to an arbitrarily chosen subset of a edges so the cardinality of the preimage of the element (p_1, p_2) with respect to the map π given by (B.2) is $\begin{pmatrix} a+b \\ b \end{pmatrix}$; this cardinality depends only on the end points x and y of the paths p_1 and p_2 so we have

$$\dim\big((x,y)\big) = \begin{pmatrix} a+b \\ b \end{pmatrix} \dim(x) \dim(y) \ . \tag{B.3}$$

For a Bratteli diagram $\mathfrak{G}$ define $D(\mathfrak{G})_a$ by

$$D(\mathfrak{G})_a := \sum_{x:\deg(x)=a} \big(\dim(x)\big)^2 \ . \tag{B.4}$$

When the Bratteli diagram is associated to an ascending chain of finite-dimensional semi-simple associative algebras, like (B.1), and the vertices correspond to irreducible representations, the number $D(\mathfrak{G})_a$ is the dimension of the algebra $\mathcal{A}_a$.

By (B.3), we have for the product

$$\begin{aligned} D(\mathfrak{G}^{(1)} \times \mathfrak{G}^{(2)})_c &= \sum_{\substack{a,b\,:\,a+b=c \\ x\,:\,deg(x)=a \\ y\,:\,deg(y)=b}} \big(\dim\big((x,y)\big)^2 = \sum_{\substack{a,b\,:\,a+b=c \\ x\,:\,deg(x)=a \\ y\,:\,deg(y)=b}} \begin{pmatrix} a+b \\ b \end{pmatrix}^2 \big(\dim(x)\big)^2 \big(\dim(y)\big)^2 \\ &= \sum_{a=0}^{c} \begin{pmatrix} c \\ a \end{pmatrix}^2 D(\mathfrak{G}^{(1)})_a \ \ D(\mathfrak{G}^{(2)})_{c-a} \ . \end{aligned} \tag{B.5}$$

B.3 Powers of Young graph

As we have seen in Section 3 (respectively, Section 6), the vertices of the Bratteli diagram for the chain (with respect to n) of the algebras $H(m,1,n)$ (respectively, the groups $G(m,1,n)$) naturally correspond to m-partitions, the level a consists of all m-partitions of a; the edges outgoing from the level a correspond to inclusions of m-partitions of a into m-partitions of $a+1$. Thus the Bratteli diagram for the chain $H(m,1,n)$ (or $G(m,1,n)$) is the m-th power of the Young graph.

1. Dimensions. We need to determine the dimensions of the vertices of the powers of the Young graph. We recall the definition of the hook length and the formula for the dimensions of the vertices of the Young graph. For a node α of a Young diagram the hook of α is the set of nodes containing α and the nodes which lie either under α in the same column or to the right of α in the same row. The hook length h_α of α is the number of nodes in the hook of α. The dimension of a representation (of a symmetric group) corresponding to a partition λ of n is given by the classical hook formula,

$$\dim(V_\lambda) = \frac{n!}{\prod\limits_{\alpha\in\lambda} h_\alpha} \ , \tag{B.6}$$

where the product $\prod\limits_{\alpha\in\lambda} h_\alpha$ means the product of the hook lengths of all nodes α of the Young diagram of shape λ.

Consider an m-partition $\lambda^{(m)} := (\lambda_1, \ldots, \lambda_m)$ such that $|\lambda^{(m)}| = n$ (we remind that $|\lambda^{(m)}| = |\lambda_1| + \cdots + |\lambda_m|$). We denote by $V_{\lambda^{(m)}}$ the irreducible representation of $H(m,1,n)$ associated with $\lambda^{(m)}$. By the generalization of (B.3) to the product of m graded graphs, the dimension of $V_{\lambda^{(m)}}$ is

$$\dim(V_{\lambda^{(m)}}) = \frac{n!}{|\lambda_1|! \ldots |\lambda_m|!} \, \frac{|\lambda_1|!}{\prod\limits_{\alpha\in\lambda_1} h_\alpha} \cdots \frac{|\lambda_m|!}{\prod\limits_{\alpha\in\lambda_m} h_\alpha} = \frac{n!}{\prod\limits_{i=1}^{m} \prod\limits_{\alpha\in\lambda_i} h_\alpha} \ , \tag{B.7}$$

Lemma 41. *We have*

$$\sum_{\lambda^{(m)}} (\dim(V_{\lambda^{(m)}}))^2 = n! m^n \ , \tag{B.8}$$

where the sum is over all m-partitions $\lambda^{(m)} = (\lambda_1, \ldots, \lambda_m)$ such that $|\lambda^{(m)}| = n$.

Proof. For $m = 1$, we know that the representations V_λ where λ is a partition of n are all the irreducible representations of the symmetric group S_n and so:

$$\sum_{\lambda} (\dim(V_\lambda))^2 = \sum_{\lambda} \left(\frac{n!}{\prod\limits_{\alpha\in\lambda} h_\alpha} \right)^2 = n! \ . \tag{B.9}$$

The proof of (B.8) is by induction on m. We have:

$$\begin{aligned}
\sum_{\lambda^{(m)}:|\lambda^{(m)}|=n} (\dim(V_{\lambda^{(m)}}))^2 &= \sum_{j=0}^{n} \left(\tfrac{n!}{(n-j)!j!}\right)^2 \sum_{\lambda^{(1)}:|\lambda^{(1)}|=j} \big(\dim(V_{\lambda^{(1)}})\big)^2 \sum_{\lambda^{(m-1)}:|\lambda^{(m-1)}|=n-j} \big(\dim(V_{\lambda^{(m-1)}})\big)^2 \\
&= \sum_{j=0}^{n} \left(\left(\tfrac{n!}{(n-j)!j!}\right)^2 \cdot j! \cdot (n-j)! \cdot (m-1)^{n-j} \right) \\
&= n! \cdot \sum_{j=0}^{n} \tfrac{n!}{(n-j)!j!} (m-1)^{n-j} = n!\, m^n \ ;
\end{aligned}$$

here $\lambda^{(1)}$ is a usual partition and $\lambda^{(m-1)}$ is an $(m-1)$-partition. In the first equality we used (B.5); in the second equality we used (B.9) and the induction hypothesis; we simplified the result in the third equality and used the binomial theorem in the fourth equality. $\square$

Remark. The m-th power of the Young graph is an m-differential poset; the formula (B.8) holds for arbitrary m-differential posets (see [36] for definitions and details).

2. Standard m-tableaux and dimensions. It is well known that the dimension of a representation of a symmetric group corresponding to some Young diagram λ equals the dimension of the corresponding vertex in the Young graph and equals the number of standard tableaux of the shape λ. It is straightforward to generalize these equalities to the cyclotomic case: the dimension of a representation of the group $G(m,1,n)$ corresponding to some Young m-diagram $\lambda^{(m)}$ equals the dimension of the corresponding vertex in the m-th power of the Young graph and equals the number of standard m-tableaux of the shape $\lambda^{(m)}$.

With the help of the Lemma 41 we check that the sum of the squares of the dimensions of the representations constructed in Subsection 4.3 (respectively, in Subsection 6.8) equals the dimension of the algebra $H(m,1,n)$ (respectively, the order of the group $G(m,1,n)$):

$$\sum_{\lambda^{(m)}} (\dim(V_{\lambda^{(m)}}))^2 = \dim(H(m,1,n)) = |G(m,1,n)| \ . \tag{B.10}$$

3. Example: square of the Young graph. Below the beginning of the Bratteli diagram for the chain of algebras $H(2,1,n)$, the square of the Young graph, is drawn. The labels on the edges correspond to the eigenvalues of the Jucys–Murphy elements of $H(2,1,n)$ (the edges going down from the level i to the level $i+1$ are labeled by the eigenvalues of the element J_{i+1}; the top vertex is situated at level 0).

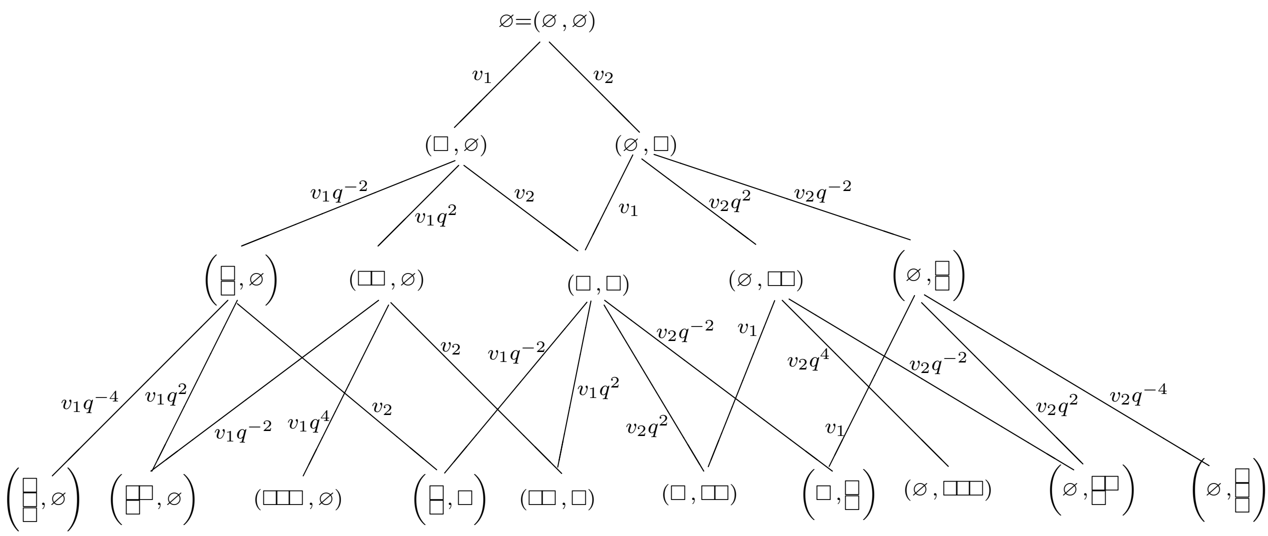

Fig. 2. *Bratteli diagram (four first levels) for $H(m,1,n)$ with $m=2$.*

Appendix C. Examples

Here we illustrate the construction of irreducible representations of the algebras $H(m,1,n)$ on several examples with $m=2$ and small n. For these examples we write down the formulas (4.8)-(4.9) and (4.16)-(4.17) from Section 4.

1. The representation of $H(2,1,2)$ corresponding to the 2-partition $(\square\,,\square)$.

The dimension of this representation is 2. We choose a basis

$$\mathcal{X}_1 := \mathcal{X}_{\left(\boxed{1},\boxed{2}\right)}\ ,\quad \mathcal{X}_2 := \mathcal{X}_{\left(\boxed{2},\boxed{1}\right)}\ .$$

The formulas (4.8)-(4.9) take the form

$$\left(\sigma_1+\tfrac{(q-q^{-1})v_2}{v_1-v_2}\right)\mathcal{X}_1=\mathcal{X}_2\left(\sigma_1+\tfrac{(q-q^{-1})v_1}{v_2-v_1}\right)\ ,\quad \left(\sigma_1+\tfrac{(q-q^{-1})v_1}{v_2-v_1}\right)\mathcal{X}_2=\mathcal{X}_1\left(\sigma_1+\tfrac{(q-q^{-1})v_2}{v_1-v_2}\right)$$

and

$$(\tau-v_1)\mathcal{X}_1=0\quad ,\qquad (\tau-v_2)\mathcal{X}_2=0\ .$$

The matrices corresponding to the action (4.16)-(4.17) of the generators of $H(2,1,2)$ in the basis above are given by:

$$\sigma_1\mapsto\begin{pmatrix} -(q-q^{-1})\frac{v_2}{v_1-v_2} & \frac{qv_1-q^{-1}v_2}{v_1-v_2} \\ \frac{qv_2-q^{-1}v_1}{v_2-v_1} & -(q-q^{-1})\frac{v_1}{v_2-v_1}\end{pmatrix}\ ,\quad \tau\mapsto \mathrm{diag}(v_1,v_2)\ . \tag{C.1}$$

The Gram matrix of the invariant ω-sesquilinear scalar product in the basis $\{\mathcal{X}_1,\mathcal{X}_2\}$ reads

$$\mathrm{diag}\left(\frac{q^{-1}v_1-qv_2}{v_1-v_2},\frac{qv_1-q^{-1}v_2}{v_1-v_2}\right)\ .$$

2. The representation of $H(2,1,3)$ corresponding to the 2-partition $\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\,,\square\right)$.

The representation has dimension 3 and we choose a basis

$$\mathcal{X}_1 := \mathcal{X}_{\left(\begin{smallmatrix}\boxed{1}\\\boxed{2}\end{smallmatrix},\boxed{3}\right)}\ ,\quad \mathcal{X}_2 := \mathcal{X}_{\left(\begin{smallmatrix}\boxed{1}\\\boxed{3}\end{smallmatrix},\boxed{2}\right)}\ ,\quad \mathcal{X}_3 := \mathcal{X}_{\left(\begin{smallmatrix}\boxed{2}\\\boxed{3}\end{smallmatrix},\boxed{1}\right)}\ .$$

The formulas (4.8)-(4.9) take the form

$$\left(\sigma_1+\tfrac{(q-q^{-1})v_1q^{-2}}{v_1-v_1q^{-2}}\right)\mathcal{X}_1=0\ ,\qquad \left(\sigma_1+\tfrac{(q-q^{-1})v_2}{v_1-v_2}\right)\mathcal{X}_2=\mathcal{X}_3\left(\sigma_1+\tfrac{(q-q^{-1})v_1}{v_2-v_1}\right)\ ,$$

$$\left(\sigma_1+\tfrac{(q-q^{-1})v_1}{v_2-v_1}\right)\mathcal{X}_3=\mathcal{X}_2\left(\sigma_1+\tfrac{(q-q^{-1})v_2}{v_1-v_2}\right)\ ,$$

$$\left(\sigma_2+\frac{(q-q^{-1})v_2}{v_1q^{-2}-v_2}\right)\mathcal{X}_1=\mathcal{X}_2\left(\sigma_2+\frac{(q-q^{-1})v_1q^{-2}}{v_2-v_1q^{-2}}\right)\ ,\quad \left(\sigma_2+\frac{(q-q^{-1})v_1q^{-2}}{v_2-v_1q^{-2}}\right)\mathcal{X}_2=\mathcal{X}_1\left(\sigma_2+\frac{(q-q^{-1})v_2}{v_1q^{-2}-v_2}\right)\ ,$$
$$\left(\sigma_2+\frac{(q-q^{-1})v_1q^{-2}}{v_1-v_1q^{-2}}\right)\mathcal{X}_3=0\ ,$$

and

$$(\tau-v_1)\mathcal{X}_1=0\ ,\quad (\tau-v_1)\mathcal{X}_2=0\ ,\quad (\tau-v_2)\mathcal{X}_3=0\ .$$

The matrices corresponding to the action (4.16)-(4.17) of the generators of $H(2,1,3)$ in the basis above are given by:

$$\sigma_1\mapsto\begin{pmatrix} -q^{-1} & 0 & 0\\ 0 & -\frac{(q-q^{-1})v_2}{v_1-v_2} & \frac{qv_1-q^{-1}v_2}{v_1-v_2}\\ 0 & \frac{qv_2-q^{-1}v_1}{v_2-v_1} & -\frac{(q-q^{-1})v_1}{v_2-v_1}\end{pmatrix}\ ,\quad \sigma_2\mapsto\begin{pmatrix} -\frac{(q-q^{-1})v_2}{v_1q^{-2}-v_2} & \frac{v_1q^{-1}-q^{-1}v_2}{v_1q^{-2}-v_2} & 0\\ \frac{qv_2-v_1q^{-3}}{v_2-v_1q^{-2}} & -\frac{(q-q^{-1})v_1q^{-2}}{v_2-v_1q^{-2}} & 0\\ 0 & 0 & -q^{-1}\end{pmatrix}$$

and

$$\tau\mapsto\mathrm{diag}(v_1,v_1,v_2)\ .$$

The Gram matrix of the invariant ω-sesquilinear scalar product in the basis $\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3\}$ reads

$$\mathrm{diag}\left(\frac{q^{-2}v_1-q^2v_2}{v_1-v_2},1,\frac{qv_1-q^{-1}v_2}{q^{-1}v_1-qv_2}\right)\ .$$

3. The representation of $H(2,1,4)$ labeled by the 2-partition $\left(\begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array}\ ,\begin{array}{|c|c|}\hline \ & \ \\\hline\end{array}\right)$.

The representation has dimension 6 and we choose a basis

$$\mathcal{X}_1:=\mathcal{X}_{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,,\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}\right)}\ ,\quad \mathcal{X}_2:=\mathcal{X}_{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,,\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}\right)}\ ,\quad \mathcal{X}_3:=\mathcal{X}_{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\,,\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\right)}\ ,\quad \mathcal{X}_4:=\mathcal{X}_{\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,,\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}\right)}\ ,$$

$$\mathcal{X}_5:=\mathcal{X}_{\left(\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\,,\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}\right)}\ ,\quad \mathcal{X}_6:=\mathcal{X}_{\left(\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\,,\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right)}\ .$$

The formulas (4.8)-(4.9) take the form

$$\left(\sigma_1+\frac{(q-q^{-1})v_1q^{-2}}{v_1-v_1q^{-2}}\right)\mathcal{X}_1=0\,,\qquad \left(\sigma_1+\frac{(q-q^{-1})v_2}{v_1-v_2}\right)\mathcal{X}_2=\mathcal{X}_4\left(\sigma_1+\frac{(q-q^{-1})v_1}{v_2-v_1}\right)\ ,$$
$$\left(\sigma_1+\frac{(q-q^{-1})v_2}{v_1-v_2}\right)\mathcal{X}_3=\mathcal{X}_5\left(\sigma_1+\frac{(q-q^{-1})v_1}{v_2-v_1}\right)\ ,\quad \left(\sigma_1+\frac{(q-q^{-1})v_1}{v_2-v_1}\right)\mathcal{X}_4=\mathcal{X}_2\left(\sigma_1+\frac{v_2}{(q-q^{-1})v_1-v_2}\right)\ ,$$
$$\left(\sigma_1+\frac{(q-q^{-1})v_1}{v_2-v_1}\right)\mathcal{X}_5=\mathcal{X}_3\left(\sigma_1+\frac{(q-q^{-1})v_2}{v_1-v_2}\right)\ ,\quad \left(\sigma_1+\frac{(q-q^{-1})v_2q^2}{v_2-v_2q^2}\right)\mathcal{X}_6=0\,,$$

$$\left(\sigma_2+\tfrac{(q-q^{-1})v_2}{v_1q^{-2}-v_2}\right)\mathcal{X}_1=\mathcal{X}_2\left(\sigma_2+\tfrac{(q-q^{-1})v_1q^{-2}}{v_2-v_1q^{-2}}\right)\,,\quad \left(\sigma_2+\tfrac{(q-q^{-1})v_1q^{-2}}{v_2-v_1q^{-2}}\right)\mathcal{X}_2=\mathcal{X}_1\left(\sigma_2+\tfrac{(q-q^{-1})v_2}{v_1q^{-2}-v_2}\right)\,,$$
$$\left(\sigma_2+\tfrac{(q-q^{-1})v_2q^2}{v_2-v_2q^2}\right)\mathcal{X}_3=0\,,\quad \left(\sigma_2+\tfrac{(q-q^{-1})v_1q^{-2}}{v_1-v_1q^{-2}}\right)\mathcal{X}_4=0\,,$$
$$\left(\sigma_2+\tfrac{(q-q^{-1})v_2q^2}{v_1-v_2q^2}\right)\mathcal{X}_5=\mathcal{X}_6\left(\sigma_2+\tfrac{(q-q^{-1})v_1}{v_2q^2-v_1}\right)\,,\quad \left(\sigma_2+\tfrac{(q-q^{-1})v_1}{v_2q^2-v_1}\right)\mathcal{X}_6=\mathcal{X}_5\left(\sigma_2+\tfrac{(q-q^{-1})v_2q^2}{v_1-v_2q^2}\right)\,,$$

$$\left(\sigma_3+\tfrac{(q-q^{-1})v_2q^2}{v_2-v_2q^2}\right)\mathcal{X}_1=0\,,\quad \left(\sigma_3+\tfrac{(q-q^{-1})v_2q^2}{v_1q^{-2}-v_2q^2}\right)\mathcal{X}_2=\mathcal{X}_3\left(\sigma_3+\tfrac{(q-q^{-1})v_1q^{-2}}{v_2q^2-v_1q^{-2}}\right)\,,$$
$$\left(\sigma_3+\tfrac{(q-q^{-1})v_1q^{-2}}{v_2q^2-v_1q^{-2}}\right)\mathcal{X}_3=\mathcal{X}_2\left(\sigma_3+\tfrac{(q-q^{-1})v_2q^2}{v_1q^{-2}-v_2q^2}\right)\,,\quad \left(\sigma_3+\tfrac{(q-q^{-1})v_2q^2}{v_1q^{-2}-v_2q^2}\right)\mathcal{X}_4=\mathcal{X}_5\left(\sigma_3+\tfrac{(q-q^{-1})v_1q^{-2}}{v_2q^2-v_1q^{-2}}\right)\,,$$
$$\left(\sigma_3+\tfrac{(q-q^{-1})v_1q^{-2}}{v_2q^2-v_1q^{-2}}\right)\mathcal{X}_5=\mathcal{X}_6\left(\sigma_3+\tfrac{(q-q^{-1})v_2q^2}{v_1q^{-2}-v_2q^2}\right)\,,\quad \left(\sigma_3+\tfrac{(q-q^{-1})v_1q^{-2}}{v_1-v_1q^{-2}}\right)\mathcal{X}_6=0\,,$$

and

$$(\tau-v_1)\mathcal{X}_1=0\ ,\quad (\tau-v_1)\mathcal{X}_2=0\ ,\quad (\tau-v_1)\mathcal{X}_3=0\ ,$$
$$(\tau-v_2)\mathcal{X}_4=0\ ,\quad (\tau-v_2)\mathcal{X}_5=0\ ,\quad (\tau-v_2)\mathcal{X}_6=0\ .$$

The matrices corresponding to the action (4.16)-(4.17) of the generators of $H(2,1,4)$ in the basis above are given by:

$$\sigma_1\mapsto\begin{pmatrix} -q^{-1} & 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{(q-q^{-1})v_2}{v_1-v_2} & 0 & \frac{qv_1-q^{-1}v_2}{v_1-v_2} & 0 & 0\\ 0 & 0 & -\frac{(q-q^{-1})v_2}{v_1-v_2} & 0 & \frac{qv_1-q^{-1}v_2}{v_1-v_2} & 0\\ 0 & \frac{qv_2-q^{-1}v_1}{v_2-v_1} & 0 & -\frac{(q-q^{-1})v_1}{v_2-v_1} & 0 & 0\\ 0 & 0 & \frac{qv_2-q^{-1}v_1}{v_2-v_1} & 0 & -\frac{(q-q^{-1})v_1}{v_2-v_1} & 0\\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix},$$

$$\sigma_2 \mapsto \begin{pmatrix} -\frac{(q-q^{-1})v_2}{v_1q^{-2}-v_2} & \frac{v_1q^{-1}-q^{-1}v_2}{v_1q^{-2}-v_2} & 0 & 0 & 0 & 0 \\ \frac{qv_2-v_1q^{-3}}{v_2-v_1q^{-2}} & -\frac{(q-q^{-1})v_1q^{-2}}{v_2-v_1q^{-2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(q-q^{-1})v_2q^2}{v_1-v_2q^2} & \frac{qv_1-v_2q}{v_1-v_2q^2} \\ 0 & 0 & 0 & 0 & \frac{v_2q^3-q^{-1}v_1}{v_2q^2-v_1} & -\frac{(q-q^{-1})v_1}{v_2q^2-v_1} \end{pmatrix} ,$$

$$\sigma_3 \mapsto \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{(q-q^{-1})v_2q^2}{v_1q^{-2}-v_2q^2} & \frac{v_1q^{-1}-v_2q}{v_1q^{-2}-v_2q^2} & 0 & 0 & 0 \\ 0 & \frac{v_2q^3-v_1q^{-3}}{v_2q^2-v_1q^{-2}} & -\frac{(q-q^{-1})v_1q^{-2}}{v_2q^2-v_1q^{-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{(q-q^{-1})v_2q^2}{v_1q^{-2}-v_2q^2} & \frac{v_1q^{-1}-v_2q}{v_1q^{-2}-v_2q^2} & 0 \\ 0 & 0 & 0 & \frac{v_2q^3-v_1q^{-3}}{v_2q^2-v_1q^{-2}} & -\frac{(q-q^{-1})v_1q^{-2}}{v_2q^2-v_1q^{-2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^{-1} \end{pmatrix} ,$$

$$\tau \mapsto \mathrm{diag}(v_1, v_1, v_1, v_2, v_2, v_2) \ .$$

The Gram matrix of the invariant ω-sesquilinear scalar product in the basis $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4, \mathcal{X}_5, \mathcal{X}_6\}$ is $\mathrm{diag}(z_1, z_2, z_3, z_4, z_5, z_6)$ where

$$z_1 = \frac{(q^{-2}v_1-q^2v_2)(q^{-3}v_1-q^3v_2)}{(v_1-v_2)(q^{-1}v_1-qv_2)} \ , \quad z_2 = \frac{q^{-3}v_1-q^3v_2}{q^{-1}v_1-qv_2} \ , \quad z_3 = 1 \ , \quad z_4 = \frac{(qv_1-q^{-1}v_2)(q^{-3}v_1-q^3v_2)}{(q^{-1}v_1-qv_2)^2} \ ,$$

$$z_5 = \frac{qv_1-q^{-1}v_2}{q^{-1}v_1-qv_2} \ , \quad z_6 = \frac{(v_1-v_2)(qv_1-q^{-1}v_2)}{(q^{-1}v_1-qv_2)(q^{-2}v_1-q^2v_2)} \ .$$

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